DESIGN OF THE
ROOSEVELT ROAD VIADUCT

BY

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ARMOUR INSTITUTE OF TECHNOLOGY

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THE PRACTICAL ASPECT OF THE SLOPE DEFLECTION METHOD AND ITS APPLICATION TO THE DESIGN OF THE ROOSEVELT ROAD VIADUCT

A THESIS

PRESENTED BY

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TO THE

PRESIDENT AND FACULTY

OF

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PREFACE.

The present work covers two distinct problems. For ease and clearness they are treated separately in what follows. The problem dealing with the practical aspects of the slope-deflection method has its proper place at the beginning. The other problem, applying the slope-deflection method to the analysis of stresses in a typical section of the Roosevelt Road Viaduct, follows the first.

In this work no claim is made to originality in working out and applying the principles of the slope-deflection method. However, the derivation of slope-deflection equations to apply to straight members with varying moments of inertia has not appeared in print before.

Works on the slope-deflection method published heretofore usually derive the fundamental equations and show their application to a theoretical case. When an application to a practical case, however, is attempted, difficulties in working out details are encountered. The present work contains an analysis and an investigation of these details (such as finding the average moment of inertia and the loading factor for a member with varying moment of inertia) as well as an application of the fundamental equations and details to a practical case.

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PART I.

PRACTICAL ASPECT OF THE SLOPE DEFLECTION METHOD.

CHAPTER I.

INTRODUCTION.

In the last few years there has been an unusual increase in reinforced concrete construction. This increase was partly due to the war with the attendant scarcity of structural steel for building purposes, but largely due to advantages inherent in the material itself.

Along with this increase came the development of the rigid frame type of buildings. In factory construction concrete roof girders rigidly connected to roof columns replace steel roof trusses. The same is true of school buildings.

It is not necessary to dwell here on the properties of concrete making for monolithic structures. The use of concrete for such structures is now considered as an established practice. Concrete as poured under ordinary conditions gives a practically monolithic structure and with a little extra precaution in the mixing, handling and pouring it, the resulting material will satisfy all assumptions of relative rigidity made in the design of rigid frames.

The development of the rigid frame type of buildings calls for greater refinement in the usual methods of analysis and design. The bending moment constants (1/8, 1/10, and 1/12) used in designing ordinary buildings cannot be properly applied to the analysis of rigid frame structures. Application of such constants would not give the proper proportions to the

members, hence the structure would not be economical. Also, since concrete is weak in tension, and is to be reinforced with steel wherever tension occurs, it becomes important for the safety of the structure to know the distribution of stress. Such distribution is not given with any degree of accuracy by the usual moment coefficients. This is the reason for the appearance in engineering periodicals of numerous articles which either work out methods of analyzing stresses in rigid frames or give short-cuts to avoid excessive computations involved in these methods.

There are two known methods of analyzing stresses in rigid frames: the method of least work and the elastic.

The least work method is based on a theorem, relating to internal work, and derived by Menabrea in 1858. The first applications of the theorem were made by A. Castigliano in 1879. In English there are only a few books on this subject. This lack of interest is in part due to the fact that the method, because of the great amount of computations required, is not popular.

The elastic method is based on the differential equations expressing deflections in terms of moments. There are several books treating this method. Among them may be mentioned Johnson, Bryan and Turneaure's Modern Framed Structures, Volume 2. This method is as complicated as the least work method as far as com-

putations are concerned.

Recently another method made its appearance. The slope-deflection method was first used for calculating secondary stresses in trusses. But later it was applied to analyzing stresses in frames due to loads and variations of temperature.

A comparison of the least work, elastic and slope deflection methods will be given in chapter 4.

CHAPTER II.

REVIEW OF THE DEVELOPMENT OF THE SLOPE DEFLECTION METHOD.

The first appearance of the method in the American press was in an article by F. C. Kunz in the Engineering News, Oct. 5, 1911, entitled "Secondary Stresses." Mr. Kunz refers to Otto Mohr as the inventor of the method. It is applied to obtain the bending moment at the joint of a member in a steel truss, when one end of the member deflects and changes its inclination with respect to the other end. Mr. Kunz indicates the derivation (analytical) of the fundamental equations for such a case. His equations are incomplete insofar as they do not take in the influence on the bending moment at the joint due to a load on the member.

In March 1915 appeared a bulletin by G. A. Maney, entitled "Secondary Stresses and Other Problems in Rigid Frames: a New Method of Solution" (University of Minnesota, Studies in Engineering #1). Mr. Maney gives a complete derivation of the fundamental equations, including the case of any system of loads on the member.

He uses "area-moments" to find deflections and slopes.* The equations derived in this bulletin are not in the form in which they are now used. However, they can be brought to the standard form by a simple transformation.

In June 1915 appeared Bulletin No. 80, University of Illinois, Engineering Experiment Station: "Wind Stresses in the Steel Frames of Office Buildings" by W. M. Wilson and G. A. Maney. (A part of this bulletin was printed in the Journal of the Western Society of Engineers, April 1915, Page 365.) The bulletin credits the theoretical work to Mr. Maney. The fundamental equations are derived (using area-moments) in the form in which they are now used, but the case of loading on the member is not worked out. It is interesting to quote the assumptions upon which the analysis is based:

1. "The connections between the columns and girders are perfectly rigid.
2. "The change in length of a member due to direct stress is equal to zero.

---

* For a complete study of "area-moments" see "Graphical Calculus", by Professor C. A. Ellis in the Journal of the Western Society of Engineers, April 1917.

3. "The length of a girder is the distance between the neutral axes of the columns which it connects; and the length of a column is the distance between the neutral axes of the girders which it connects.

4. "The deflection of a member due to internal shearing stresses is equal to zero."

The name of the method (slope-deflection) appears in this bulletin for the first time.

"The Concrete Engineer's Handbook" by G. A. Hool and N. C. Johnson has two chapters on the slope-deflection method. The fundamental equations are derived by means of "area-moments" in the first chapter. But the derivation is not rigid. For instance: on page 412 of the first edition are given two equations:

$$-(d-\theta_A l) = \frac{l^2}{EI}\left(\frac{M_{AB}}{3}+\frac{M_{BA}}{6}\right)$$

$$(\theta_A-\theta_B) = \frac{l}{2EI}\left(M_{AB}+M_{BA}\right);$$

from which the authors claim that

$$M_{BA} = 2EK\left(2\theta_B+\theta_A-\frac{3d}{l}\right), \quad \text{where } K=\frac{I}{l}$$

may be obtained. But by solving these two equations simultaneously $M_{BA} = -2EK\left(2\theta_B+\theta_A-\frac{3d}{l}\right).$

is obtained.

In a subsequent chapter of the book, the authors apply the slope-deflection method to a viaduct frame. The application is theoretical, giving moments at the joints of the frame for various conditions of load and for temperature variations.

In May 1919 appeared Bulletin #108 of the University of Illinois Engineering Experiment Station: "Analysis of Statically Indeterminate Structures by the Slope-Deflection Method" by W. M. Wilson, F. E. Richart and C. Weiss. The fundamental equations are derived by means of "area-moments". They are first applied to continuous beams and the slope-deflection form of the three moment theorem is worked out. Then they are applied to a portal frame under various loads and various end conditions. Then an application is made to a 2 span frame.

The assumptions upon which the applications are made are as follows (see Page 28 of bulletin):

1. The connections are perfectly rigid.
2. The length of a member is not changed by axial stress.
3. The shearing deformation is zero.

The discussion of the assumptions is of interest.

In the Am. Soc. Civ. Eng. Proc. of August, 1919, Mr. G. A. Maney discusses the application of the slope deflection method to curved beams and arches. The formulas derived are complicated, especially for the case of loading on the member.

CHAPTER III.

DERIVATION OF THE FUNDAMENTAL EQUATIONS.

In the works reviewed in Chapter **II**, the methods used for deriving the fundamental equations are the analytical and the "area-moment". Both methods possess their advantages and disadvantages. The analytical method, while involving tedious transformations, possesses the advantage of simplicity in beginning the derivation and ease in handling signs of quantities. The "area-moment" method has less algebraic transformations, but is more complicated as regards signs.

In view of these facts, and also because previous works have all used the "area-moment" method (with the exception of Professor Kunz), the analytical method for the derivation of the fundamental equations is adopted in this work.

As a preliminary it is necessary to find an expression for the moment at any point of a member subjected to end-moments, but without superimposed loading.

1. MOMENT AT ANY POINT

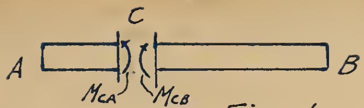


Fig. 1.

Assumptions: Moments causing clockwise rotation are positive. M means moment at any section; M_X means moment at section X; M_{XA} means moment at section x acting on part of section toward A. Thus, in Fig. 1, M_A means moment at A; M_{CA} means moment on left hand part of section; M_{CB} means moment on right hand part of section.

Let AB (Fig. 1) be a member of a frame in equilibrium. Cut a section at any point C and let M_{CA} and M_{CB} be moments required to hold in equilibrium left hand and right hand parts respectively. Then $M_{CA} = -M_{CB}$.

For plotting moment diagrams moments acting on right hand part of section will be used. When positive the moment will be laid off below the axis. This will mean that moments are plotted on the tension side.

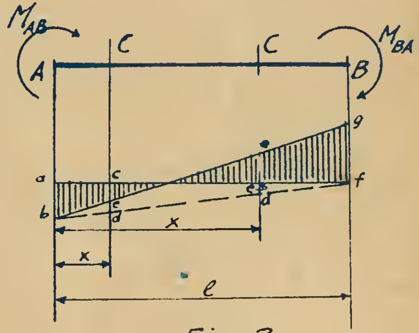


Fig. 2

Case a. M_A and M_B have same direction. See Fig. 2. Let AB be part of a frame and M_{AB} and M_{BA} moments at A and B caused by restraint of the frame. M_{AB} is positive and acts on the right hand part, it is therefore plotted below the base line a-f : $M_{AB} = ab$. M_{BA} is positive, but acts on the left hand part. The moment

at B for the right hand part (which is supplied by the rest of the frame) is negative and equals $-M_{BA}$; it is plotted above the base line as f-g. The moment M_{CB} at any section C equals $c\text{-}e = cd - ed$. But c-d equals $a\text{-}b \times \left(\frac{l-x}{l}\right)$ or $M_{AB}\left(\frac{l-x}{l}\right)$ and $e\text{-}d = g\text{-}f \cdot \frac{x}{l} = M_{BA}\frac{x}{l}$. Hence $M_{CB} = M_{AB}\left(\frac{l-x}{l}\right) - M_{BA}\frac{x}{l}$.

Case b. M_{AB} and M_{BA} have different directions.

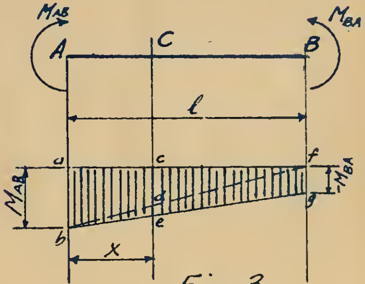


Fig. 3

See Fig. 3. M_{AB} is positive and acts on the right hand part of the section, it is plotted below the base line as a-b. M_{BA} is negative and acts on the left hand part of section. The moment at B for the right hand part (which is supplied by the rest of the frame) is positive and therefore plotted below the base line as f g. In plotting the numerical value is used. The numerical value of M_{BA} is $-M_{BA}$, since M_{BA} is a negative quantity, as is shown by direction of arrow in Fig.3.*

The moment M_{CB} at any section C equals c e.

$$c\,e = c\,d + de = M_{AB}\left(\frac{l-x}{l}\right) + \frac{-M_{BA}x}{l} = M_{AB}\left(\frac{l-x}{l}\right) - M_{BA}\frac{x}{l}.$$

---

* Showing the direction of rotation means assuming the sign of the moment. The correctness or incorrectness of the assumption will be known when the numerical value of the moment is obtained.

Hence $M_{AB}\frac{(l-x)}{l} - M_{BA}\frac{x}{l}$ is the general expression of the moment at a section distant x from A, when M_{AB} and M_{BA} are given.

2. FUNDAMENTAL EQUATIONS FOR MEMBER RESTRAINED AT BOTH ENDS AND UNIFORMLY LOADED.

Case a. No deflection of A relative to B.

See Figs. 4 and 5. Let AB be part of a frame subjected to moments M_{AB} and M_{BA} due to the restraint caused by the frame. Let AB be loaded with a uniform load of w lbs. per ft.

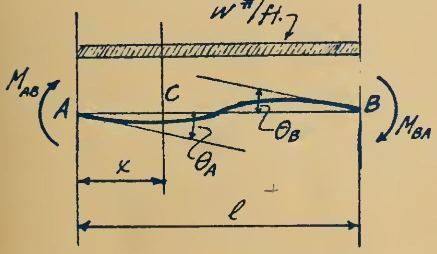


Fig. 4

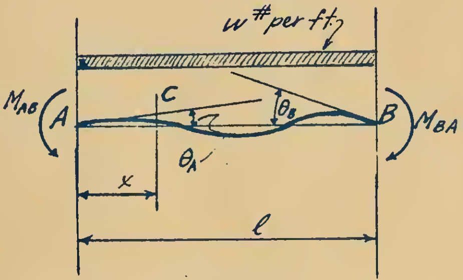


Fig. 5

The heavy line AB shows the deformed elastic line (the curvature is exaggerated). Angles θ_A and θ_B are made by tangents to the elastic curve at A and B respectively with the initial position of the elastic line — the neutral axis of the member.

Assume origin of coördinates at A. x is positive

when measured toward the right; y is positive when measured downward. An angle is positive when formed by a line moving in a clockwise direction from its initial position; thus, in Fig. 4 θ_A and θ_B are positive, in Fig. 5 θ_A is negative and θ_B is positive.

The equation of the elastic curve is $EI\frac{d^2y}{dx^2} = -M$.* The moment at any section C distant x from A is $M_x = M_{AB}\frac{(\ell - x)}{\ell} - M_{BA}\frac{x}{\ell} + \frac{w\ell x}{2} - \frac{wx^2}{2}$. Hence

$$EI\frac{d^2y}{dx^2} = -M_{AB} + M_{AB}\frac{x}{\ell} + M_{BA}\frac{x}{\ell} - \frac{w\ell x}{2} + \frac{wx^2}{2}$$

Integrate,

$$EI\frac{dy}{dx} = -M_{AB}\,x + M_{AB}\frac{x^2}{2\ell} + M_{BA}\frac{x^2}{2\ell} - \frac{w\ell x^2}{4} + \frac{wx^3}{6} + C_1 \text{ ——— } (1)$$

When $x = 0$; $\frac{dy}{dx} = \theta_A$

$$\therefore\ EI\,\theta_A = C_1$$

When $x = \ell$; $\frac{dy}{dx} = \theta_B$. Substitute in (1)

$$EI\,\theta_B = -M_{AB}\,\ell + M_{AB}\frac{\ell}{2} + M_{BA}\frac{\ell}{2} - \frac{w\ell^3}{4} + \frac{w\ell^3}{6} + EI\,\theta_A$$

$$\text{or}\quad EI(\theta_B - \theta_A) = -M_{AB}\frac{\ell}{2} + M_{BA}\frac{\ell}{2} - \frac{w\ell^3}{12}$$

$$\text{or}\quad \frac{2}{\ell}EI(\theta_B - \theta_A) = -M_{AB} + M_{BA} - \frac{w\ell^2}{6} \text{ ——————— } (2)$$

Integrate (1), remembering that $C_1 = EI\theta_A$,

$$EIy = -M_{AB}\frac{x^2}{2} + M_{AB}\frac{x^3}{6\ell} + M_{BA}\frac{x^3}{6\ell} - \frac{w\ell x^3}{12} + \frac{wx^4}{24} + EI\,\theta_A\,x + C_2 \quad (3a)$$

When $x = 0$, $y = 0$ $\therefore\ C_2 = 0$

When $x = \ell$, $y = 0$

$$\therefore\quad 0 = -M_{AB}\frac{\ell^2}{2} + M_{AB}\frac{\ell^2}{6} + M_{BA}\frac{\ell^2}{6} - \frac{w\ell^4}{12} + \frac{w\ell^4}{24} + EI\,\theta_A\,\ell$$

$$\text{or}\quad EI\,\theta_A\,\ell = M_{AB}\frac{\ell^2}{3} - M_{BA}\frac{\ell^2}{6} + \frac{w\ell^4}{24}$$

$$\text{or}\quad \frac{6}{\ell}EI\,\theta_A = 2M_{AB} - M_{BA} + \frac{w\ell^2}{4} \text{ ——————— } (3)$$

---

* See A. H. Heller, Stresses in Structures, p. 128.

Solving (2) and (3) simultaneously:

$$\frac{2}{l} EI(\theta_B - \theta_A) = -M_{AB} + M_{BA} - \frac{wl^2}{6} \quad \text{———} \quad (2)$$

$$\frac{6}{l} EI(\theta_A) = 2M_{AB} - M_{BA} + \frac{wl^2}{4} \quad \text{———} \quad (3)$$

$$\frac{2}{l} EI(\theta_B - \theta_A + 3\theta_A) = M_{AB} + \frac{wl^2}{12}$$

or $$\underline{\underline{M_{AB} = \frac{2EI}{l}(2\theta_A + \theta_B) - \frac{wl^2}{12}}} \quad \text{———} \quad (4)$$

Substitute (4) in (2)

$$\frac{2}{l} EI(\theta_B - \theta_A) = -\frac{2}{l} EI(2\theta_A + \theta_B) + \frac{wl^2}{12} + M_{BA} - \frac{wl^2}{6}$$

$$\frac{2}{l} EI(\theta_B - \theta_A + 2\theta_A + \theta_B) = M_{BA} - \frac{wl^2}{12}$$

or $$\underline{\underline{M_{BA} = \frac{2EI}{l}(\theta_A + 2\theta_B) + \frac{wl^2}{12}}} \quad \text{———} \quad (5)$$

(4) and (5) are the fundamental equations for the case of no deflection of one end relative to the other and of a uniform load of w #per ft.

Case b. Deflection d.

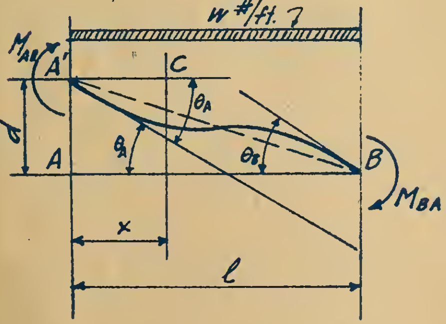


Fig. 6

Let AB be the original and A'B the final position of the neutral axis, A having deflected a distance d. The notation is the same as before. Origin of coördinates at A.

The equation of the elastic curve and the integrations are the same

as for the case of no deflections. The constants in the first integration are also the same. Hence equation (2) $\frac{2}{l} EI(\theta_B - \theta_A) = -M_{AB} + M_{BA} - \frac{wl^2}{6}$ may be used here.

Constants for second integration: When $x=0$, $y=-d$ (negative, because upward) $\therefore -EId = C_2$

When $x=l$, $y=0$. Substitute in (3a)

$$0 = -M_{AB}\frac{l^2}{2} + M_{AB}\frac{l^2}{6} + M_{BA}\frac{l^2}{6} - \frac{wl^4}{12} + \frac{wl^4}{24} + EI\theta_A l - EId$$

$$EI(\theta_A l - d) = M_{AB}\frac{l^2}{3} - M_{BA}\frac{l^2}{6} + \frac{wl^4}{24}.$$

Multiply thru by $\frac{6}{l^2}$

$$\frac{6EI}{l}\left(\theta_A - \frac{d}{l}\right) = 2M_{AB} - M_{BA} + \frac{wl^2}{4} \qquad (6)$$

Solving (2) and (6) simultaneously:

$$\frac{2}{l}EI(\theta_B - \theta_A) = -M_{AB} + M_{BA} - \frac{wl^2}{6} \qquad (2)$$

$$\frac{6}{l}EI\left(\theta_A - \frac{d}{l}\right) = 2M_{AB} - M_{BA} + \frac{wl^2}{4} \qquad (6)$$

$$\frac{2}{l}EI\left(\theta_B - \theta_A + 3\theta_A - \frac{3d}{l}\right) = M_{AB} + \frac{wl^2}{12}$$

or $$\underline{\underline{M_{AB} = \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3d}{l}\right) - \frac{wl^2}{12}}} \qquad (7)$$

Substitute (7) in (2)

$$\frac{2}{l}EI(\theta_B - \theta_A) = -\frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3d}{l}\right) + \frac{wl^2}{12} + M_{BA} - \frac{wl^2}{6}$$

$$\frac{2}{l}EI\left(\theta_B - \theta_A + 2\theta_A + \theta_B - \frac{3d}{l}\right) = M_{BA} - \frac{wl^2}{12}$$

or $$\underline{\underline{M_{BA} = \frac{2EI}{l}\left(\theta_A + 2\theta_B - \frac{3d}{l}\right) + \frac{wl^2}{12}}} \qquad (8)$$

(7) and (8) are the general fundamental equations for the case of a uniform load of w lbs. per ft.

3. FUNDAMENTAL EQUATIONS
FOR MEMBER RESTRAINED AT BOTH ENDS BUT NOT LOADED.

In equations (7) and (8) let $w = 0$, then

$$M_{AB} = \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3d}{l}\right)$$

$$M_{BA} = \frac{2EI}{l}\left(\theta_A + 2\theta_B - \frac{3d}{l}\right)$$

4. FUNDAMENTAL EQUATIONS
FOR MEMBER RESTRAINED AT BOTH ENDS WITH CONCENTRATED LOAD P AT ANY POINT.

Notation same as before.

$$M_x = M_{AB}\frac{(l-x)}{l} - M_{BA}\frac{x}{l} + P(l-a)\frac{x}{l}$$

for any point to the left of P and

$$= M_{AB}\frac{(l-x)}{l} - M_{BA}\frac{x}{l} + P(l-a)\frac{x}{l} - P(x-a)$$

for any point to the right of P.

$\frac{P(l-a)}{l}$

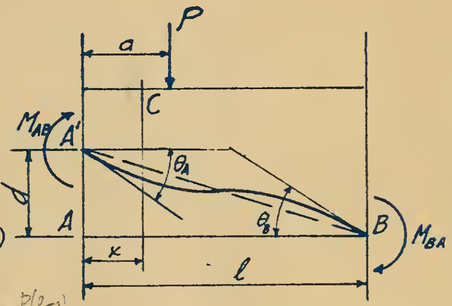


Fig. 7

Curve to left of Load.

$$EI\frac{d^2y}{dx^2} = -M_{AB} + M_{AB}\frac{x}{l} + M_{BA}\frac{x}{l} - \frac{P(l-a)x}{l}$$

Integrate

$$EI\frac{dy}{dx} = -M_{AB}x + M_{AB}\frac{x^2}{2l} + M_{BA}\frac{x^2}{2l} - P\frac{(l-a)x^2}{2l} + C_1$$

When $x = 0$ $\frac{dy}{dx} = \theta_A$ $\therefore$ $EI\theta_A = C_1$

When $x = a$

$$EI\frac{dy}{dx} = -M_{AB}a + M_{AB}\frac{a^2}{2l} + M_{BA}\frac{a^2}{2l} - P\frac{(l-a)a^2}{2l} + EI\theta_A \text{ ——— } (9)$$

Integrate again

$$EIy = -M_{AB}\frac{x^2}{2} + M_{AB}\frac{x^3}{6l} + M_{BA}\frac{x^3}{6l} - P(l-a)\frac{x^3}{6l} + EI\theta_A x + C_2$$

When $x=0$, $y=-d$ $\therefore$ $-EId = C_2$

When $x=a$

$$EIy = -M_{AB}\frac{a^2}{2} + M_{AB}\frac{a^3}{6l} + M_{BA}\frac{a^3}{6l} - P(l-a)\frac{a^3}{6l} + EI\theta_A a - EId \text{ ——— } (10)$$

Curve to right of Load.

$$EI\frac{d^2y}{dx^2} = -M_{AB} + M_{AB}\frac{x}{l} + M_{BA}\frac{x}{l} - P(l-a)\frac{x}{l} + Px - Pa$$

Integrate

$$EI\frac{dy}{dx} = -M_{AB}x + M_{AB}\frac{x^2}{2l} + M_{BA}\frac{x^2}{2l} - P(l-a)\frac{x^2}{2l} + \frac{Px^2}{2} - Pax + C_3 \text{ ——— } (11a)$$

When $x=l$, $\frac{dy}{dx} = \theta_B$

$$EI\theta_B = -M_{AB}l + M_{AB}\frac{l}{2} + M_{BA}\frac{l}{2} - P(l-a)\frac{l}{2} + \frac{Pl^2}{2} - Pal + C_3$$

$$= -M_{AB}\frac{l}{2} + M_{BA}\frac{l}{2} - \frac{Pal}{2} + C_3$$

or $C_3 = EI\theta_B + M_{AB}\frac{l}{2} - M_{BA}\frac{l}{2} + \frac{Pal}{2}$ ——— (11)

When $x=a$

$$EI\frac{dy}{dx} = -M_{AB}a + M_{AB}\frac{a^2}{2l} + M_{BA}\frac{a^2}{2l} - P(l-a)\frac{a^2}{2l} - \frac{Pa^2}{2} + EI\theta_B + M_{AB}\frac{l}{2} - M_{BA}\frac{l}{2} + \frac{Pal}{2} \quad (12)$$

Substitute value of C_3 in $EI\frac{dy}{dx}$ (11a)

$$EI\frac{dy}{dx} = -M_{AB}x + M_{AB}\frac{x^2}{2l} + M_{BA}\frac{x^2}{2l} - P(l-a)\frac{x^2}{2l} + \frac{Px^2}{2} - Pax + EI\theta_B + M_{AB}\frac{l}{2} - M_{BA}\frac{l}{2} + \frac{Pal}{2} \quad (13)$$

Integrate (13)

$$EIy = -M_{AB}\frac{x^2}{2} + M_{AB}\frac{x^3}{6l} + M_{BA}\frac{x^3}{6l} - P(l-a)\frac{x^3}{6l} + \frac{Px^3}{6} - \frac{Pax^2}{2} + EI\theta_B x + M_{AB}\frac{lx}{2}$$
$$- M_{BA}\frac{lx}{2} + \frac{Palx}{2} + C_4 \quad (13a)$$

When $x=l$, $y=0$

$$\therefore \quad 0 = -M_{AB}\frac{l^2}{2} + M_{AB}\frac{l^2}{6} + M_{BA}\frac{l^2}{6} - P(l-a)\frac{l^2}{6} + \frac{Pl^3}{6} - \frac{Pal^2}{2} + EI\theta_B l + M_{AB}\frac{l^2}{2}$$
$$- M_{BA}\frac{l^2}{2} + \frac{Pal^2}{2} + C_4$$

$$C_4 = -M_{AB}\frac{l^2}{6} + M_{BA}\frac{l^2}{3} - \frac{Pal^2}{6} - EI\theta_B l \text{ ——— } (14)$$

When $x=a$ (Substituting also value for C_4)

$$EIy = -M_{AB}\frac{a^2}{2} + M_{AB}\frac{a^3}{6l} + M_{BA}\frac{a^3}{6l} - P(l-a)\frac{a^3}{6l} + \frac{Pa^3}{6} - \frac{Pa^3}{2} + EI\theta_B a + M_{AB}\frac{la}{2}$$
$$- M_{BA}\frac{la}{2} + \frac{Pa^2l}{2} - M_{AB}\frac{l^2}{6} + M_{BA}\frac{l^2}{3} - \frac{Pal^2}{6} - EI\theta_B l \text{ ——— } (15)$$

Under the load the two elastic curves have the same tangent. Hence (9) = (12)

$$-M_{AB}a + M_{AB}\frac{a^2}{2\ell} + M_{BA}\frac{a^2}{2\ell} - P(\ell-a)\frac{a^2}{2\ell} + EI\theta_A = -M_{AB}a + M_{AB}\frac{a^2}{2\ell}$$
$$+M_{BA}\frac{a^2}{2\ell} - P(\ell-a)\frac{a^2}{2\ell} - \frac{Pa^2}{2} + EI\theta_B + M_{AB}\frac{\ell}{2} - M_{BA}\frac{\ell}{2} + Pa\frac{\ell}{2}$$

or $EI(\theta_A-\theta_B) = (M_{AB}-M_{BA})\frac{\ell}{2} + \frac{Pa\ell}{2} - \frac{Pa^2}{2}$

or $\frac{2}{\ell}EI(\theta_A-\theta_B) = M_{AB} - M_{BA} + Pa - \frac{Pa^2}{\ell}$ ———————— (16)

Under the load the two elastic curves have the same deflection. Hence (10) = (15)

$$-M_{AB}\frac{a^2}{2} + M_{BA}\frac{a^3}{6\ell} + M_{AB}\frac{a^3}{6\ell} - P(\ell-a)\frac{a^3}{6\ell} + EI\theta_A a - EId = -M_{AB}\frac{a^2}{2}$$
$$+M_{AB}\frac{a^3}{6\ell} + M_{BA}\frac{a^3}{6\ell} - P(\ell-a)\frac{a^3}{6\ell} - \frac{Pa^3}{3} + EI\theta_B a + M_{AB}\frac{\ell a}{2} - M_{BA}\frac{\ell a}{2} + \frac{Pa^2\ell}{2}$$
$$-M_{AB}\frac{\ell^2}{6} + M_{BA}\frac{\ell^2}{3} - \frac{Pa\ell^2}{6} - EI\theta_B\ell$$

or $EI(\theta_A a - d - \theta_B a + \theta_B\ell) = M_{AB}\frac{\ell a}{2} - M_{AB}\frac{\ell^2}{6} - M_{BA}\frac{\ell a}{2} + M_{BA}\frac{\ell^2}{3} - \frac{Pa^3}{3} + \frac{Pa^2\ell}{2} - \frac{Pa\ell^2}{6}$

$$= M_{AB}\frac{\ell a}{2} - M_{AB}\frac{\ell^2}{6} + M_{BA}\left(\frac{\ell}{3}-\frac{a}{2}\right)\ell - \frac{Pa^3}{3} + \frac{Pa^2\ell}{2} - \frac{Pa\ell^2}{6}$$

Multiply thru by $\frac{6}{\ell}$

$\frac{6EI}{\ell}(\theta_A a - d - \theta_B a + \theta_B\ell) = 3M_{AB}a - M_{AB}\ell + M_{BA}(2\ell-3a) - \frac{2Pa^3}{\ell} + 3Pa^2 - Pa\ell$ (17)

Multiply (16) by $2\ell-3a$ and add to (17)

$$\frac{2EI}{\ell}(3\theta_A a - 3d - 3\theta_B a + 3\theta_B\ell + 2\ell\theta_A - 3a\theta_A - 2\ell\theta_B + 3a\theta_B) = 3M_{AB}a - M_{AB}\ell$$
$$+M_{AB}2\ell - 3M_{AB}a + M_{BA}(2\ell-3a) - M_{BA}(2\ell-3a) - \frac{2Pa^3}{\ell} + 3Pa^2 - Pa\ell$$
$$+2Pa\ell - 2Pa^2 - 3Pa^2 + \frac{3Pa^3}{\ell}$$

or $\frac{2EI}{\ell}(2\ell\theta_A + \theta_B\ell - 3d) = M_{AB}\ell + \frac{Pa^3}{\ell} - 2Pa^2 + Pa\ell$

$$2EI\left(2\theta_A + \theta_B - \frac{3d}{\ell}\right) = M_{AB}\ell + \frac{Pa}{\ell}(a^2 - 2a\ell + \ell^2)$$
$$= M_{AB}\ell + \frac{Pa(a-\ell)^2}{\ell}$$

$$\therefore\ M_{AB} = \frac{2EI}{\ell}\left(2\theta_A + \theta_B - \frac{3d}{\ell}\right) - \frac{Pa(\ell-a)^2}{\ell^2} \qquad \text{———— (18)}$$

Substitute (18) in (16)

$$\frac{2}{\ell}EI(\theta_A-\theta_B) = \frac{2EI}{\ell}\left(2\theta_A + \theta_B - \frac{3d}{\ell}\right) - \frac{Pa(\ell-a)^2}{\ell^2} + Pa - \frac{Pa^2}{\ell} - M_{BA}$$

$$\frac{2EI}{l}\left(\theta_A - \theta_B - 2\theta_A - \theta_B + \frac{3d}{l}\right) = -M_{BA} - \frac{Pa}{l^2}\left(l^2 - 2al + a^2 - l^2 + al\right)$$

$$\frac{2EI}{l}\left(-\theta_A - 2\theta_B + \frac{3d}{l}\right) = -M_{BA} - \frac{Pa}{l^2}\left(a^2 - al\right)$$

$$= -M_{BA} + \frac{Pa^2}{l^2}(l-a)$$

or $$\underline{\underline{M_{BA} = \frac{2EI}{l}\left(\theta_A + 2\theta_B - \frac{3d}{l}\right) + \frac{Pa^2(l-a)}{l}}} \qquad \text{————————} \quad (19)$$

(18) and (19) are the fundamental equations for the case of load P distant a from the support A.

5. FUNDAMENTAL EQUATIONS FOR MEMBER RESTRAINED AT BOTH ENDS ANY SYSTEM OF LOADING

Take the case of any system of loads on AB. These loads may be considered as several concentrated loads, each acting at a certain distance from the support A. By going thru the same steps as for a single load P distant a from support A, it can be found that the fundamental equations are:

$$M_{AB} = \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3d}{l}\right) - \Sigma P \frac{a(l-a)^2}{l^2}$$

$$M_{BA} = \frac{2EI}{l}\left(2\theta_B + \theta_A - \frac{3d}{l}\right) + \Sigma \frac{Pa^2(l-a)}{l^2},$$ where the terms $\Sigma \frac{Pa(l-a)^2}{l^2}$ and $\Sigma \frac{Pa^2(l-a)}{l^2}$ stand for the summation of factors due to each load separately and where P stands for that load and "a" for its distance from the support A.

A slightly different and simpler form of these equations is given on the next page.

6. THE LOADING FACTOR

From the cases previously considered it is seen that the moments M_{AB} and M_{BA} consist each of two terms. The first is constant for all loading and depends on the restraint caused by the frame. The other term depends upon the load; it is called the loading factor, C (the term C uses the same scheme for subscripts as M).

Therefore the fundamental equations for any system of loads are :

$$M_{AB} = 2\frac{EI}{\ell}\left(2\theta_A+\theta_B-\frac{3d}{\ell}\right) + C_{AB}$$

$$M_{BA} = \frac{2EI}{\ell}\left(2\theta_B+\theta_A-\frac{3d}{\ell}\right) + C_{BA}.$$

Take a beam AB fixed at both ends and loaded with any system of vertical loads. The moments at the supports will be those stated above. But in a fixed beam the tangents to the elastic curve at the supports are equal to zero; also there is no deflection of one end relative to the other. That is, θ_A, θ_B, d are equal zero; and the first term of the equation equals zero.

Hence for a fixed beam,

$$M_{AB} = +C_{AB}$$

$$M_{BA} = +C_{BA}.$$

In other words, the loading factor of a member is equal to the moment at the corresponding end of a beam fixed at both ends and having the same span and loading as the member.

The sign of the loading factor is the same as for moments; that is positive when causing clockwise rotation.

The loading factors for a uniform load and for a load P distant a from left hand support were worked out in connection with the fundamental equations.

The loading factors for a few more cases of loading are worked out here.

Case a. A triangular load, zero at A and $\frac{2W}{l}$ at B, so that the total load $= W$. The load at any point distant x from A $= \frac{2Wx}{l^2}$. See Fig. 8. The loading factor due to $\frac{2Wx}{l^2}$ is $d(C_{AB}) = -\frac{2Wx}{l^2}dx.\frac{x(l-x)^2}{l^2}$ (The loading factor for a load P distant a from A, $C_{AB} = -\frac{Pa(l-a)^2}{l^2}$).

Hence $C_{AB} = -\int_0^l \frac{2Wx^2(l-x)^2}{l^4}dx = -\frac{2W}{l^4}\left(\int_0^l x^2l^2dx - \int_0^l 2lx^3dx + \int_0^l x^4dx\right) = -\frac{2W}{l^4}\left(\frac{l^5}{3} - \frac{l^5}{2} + \frac{l^5}{5}\right) = -\frac{Wl}{15}$.

Also $d(C_{BA}) = \frac{2Wx\,dx}{l^2}.\frac{x^2(l-x)}{l^2}$; since C_{BA} for a load P distant a from A $= \frac{Pa^2(l-a)}{l^2}$.

Hence $C_{BA} = \int_0^l \frac{2Wx^3(l-x)}{l^4}dx = \frac{2W}{l^4}\left(\int_0^l x^3l\,dx - \int_0^l x^4dx\right)$
$= \frac{2W}{l^4}\left(\frac{l^5}{4} - \frac{l^5}{5}\right) = \frac{Wl}{10}$.

∴ For Case a (Fig. 8)

$C_{AB} = -\frac{Wl}{15}$; $C_{BA} = +\frac{Wl}{10}$.

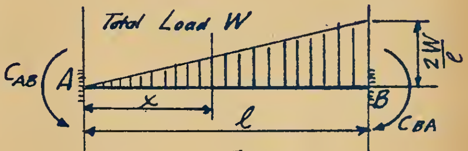


Fig. 8

Case b. Load as shown in Fig. 9. The load at any point distant x from A $= \frac{2W}{(l-a)^2}(x-a)$

$d(C_{AB}) = -\frac{2W(x-a)}{(l-a)^2}\,x(l-x)^2\frac{dx}{l^2}$

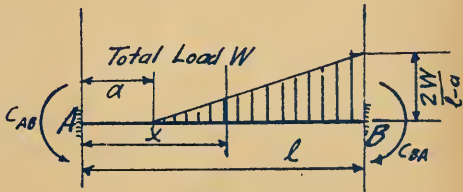


Fig. 9

$$C_{AB} = -\int_a^\ell \frac{2Wx(x-a)(\ell-x)^2dx}{(\ell-a)^2\,\ell^2} = -\frac{2W}{\ell^2(\ell-a)^2}\Big[\int_a^\ell(x^2\ell^2 - 2\ell x^3 + x^4 - \ell^2xa + 2\ell x^2a$$
$$-x^3a)\,dx = -\frac{2W}{\ell^2(\ell-a)^2}\left(\frac{\ell^2x^3}{3} - \frac{\ell x^4}{2} + \frac{x^5}{5} - \frac{\ell^2ax^2}{2} + \frac{2}{3}\ell a x^3 - \frac{ax^4}{4}\right)\Big/_a^\ell$$
$$= -\frac{2W}{\ell^2(\ell-a)^2}\left(\frac{\ell^5}{3} - \frac{\ell^5}{2} + \frac{\ell^5}{5} - \frac{\ell^4a}{2} + \frac{2}{3}\ell^4a - \frac{\ell^4a}{4} - \frac{\ell^2a^3}{3} + \frac{2\ell a^4}{4} - \frac{a^5}{5} + \frac{\ell^2a^3}{2} - \frac{2}{3}\ell a^4 + \frac{a^5}{4}\right)$$
$$= -\frac{2W}{\ell^2(\ell-a)^2}\left(\frac{\ell^5}{30} - \frac{\ell^4a}{12} + \frac{\ell^2a^3}{6} - \frac{\ell a^4}{6} + \frac{a^5}{20}\right) = \frac{-W}{30\ell^2(\ell-a)^2}(2\ell^5 - 5\ell^4a + 10\ell^2a^3 - 10\ell a^4 + 3a^5)$$
$$= \frac{-W}{30\ell^2(\ell-a)^2}(\ell-a)^4(2\ell+3a) = \frac{-W}{30\ell^2}(\ell-a)^2(2\ell+3a)$$

$$d(C_{BA}) = \frac{2W(x-a)}{(\ell-a)^2}\frac{x^2(\ell-x)dx}{\ell^2}.$$
$$C_{BA} = \frac{2W}{\ell^2(\ell-a)^2}\int_a^\ell x^2(x-a)(\ell-x)\,dx = \frac{2W}{\ell^2(\ell-a)^2}\int_a^\ell(x^3\ell - x^4 - a\ell x^2 + ax^3)dx$$
$$= \frac{2W}{\ell^2(\ell-a)^2}\left(\frac{x^4\ell}{4} - \frac{x^5}{5} - \frac{a\ell x^3}{3} + \frac{ax^4}{4}\right)\Big/_a^\ell = \frac{2W}{\ell^2(\ell-a)^2}\left(\frac{\ell^5}{4} - \frac{\ell^5}{5} - \frac{a\ell^4}{3} + \frac{a\ell^4}{4} - \frac{a^4\ell}{4} + \frac{a^5}{5} + \frac{a^4\ell}{3} - \frac{a^5}{4}\right)$$
$$= \frac{2W}{\ell^2(\ell-a)^2}\left(\frac{\ell^5}{20} - \frac{a\ell^4}{12} + \frac{\ell a^4}{12} - \frac{a^5}{20}\right) = \frac{W}{30\ell^2(\ell-a)^2}(3\ell^5 - 5a\ell^4 + 5a^4\ell - 3a^5)$$
$$= \frac{W(\ell-a)^3(3\ell^2+4\ell a+3a^2)}{30\ell^2(\ell-a)^2} = \frac{W}{30\ell^2}(3\ell^2+4\ell a+3a^2)(\ell-a)$$

$\therefore$ For Case b, Fig. 9

$$C_{AB} = -\frac{W}{30\ell^2}(\ell-a)^2(2\ell+3a)\quad;\quad C_{BA} = +\frac{W}{30\ell^2}(\ell-a)(3\ell^2+4\ell a+3a^2)$$

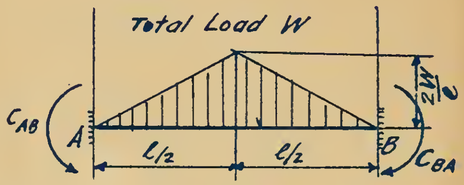


Fig. 10

Case c. Load as shown in Fig. 10. Consider the left hand half of the load separately from the right hand half. On the left hand half the load at any point distant x from A

$$= \frac{2Wx}{\ell\,\ell/2} = \frac{4Wx}{\ell^2}. \qquad d(C_{AB}) = -\frac{4Wx}{\ell^2}\frac{x(\ell-x)^2dx}{\ell^2}$$
$$C_{AB} = -\frac{4W}{\ell^4}\int_0^{\ell/2}(x^2\ell^2 - 2x^3\ell + x^4)\,dx = -\frac{4W}{\ell^4}\left(\frac{\ell^2x^3}{3} - \frac{\ell x^4}{2} + \frac{x^5}{5}\right)\Big/_0^{\ell/2}$$
$$= -\frac{4W}{\ell^4}\left(\frac{\ell^5}{24} - \frac{\ell^5}{32} + \frac{\ell^5}{160}\right) = -\frac{4W\ell}{60} = -\frac{2W\ell}{30} = -\frac{W\ell}{15}$$
$$d(C_{BA}) = \frac{4Wx}{\ell^2}\frac{x^2(\ell-x)dx}{\ell^2}$$
$$C_{BA} = \frac{4W}{\ell^4}\int_0^{\ell/2}(x^3\ell - x^4)dx = \frac{4W}{\ell^4}\left(\frac{x^4\ell}{4} - \frac{x^5}{5}\right)\Big/_0^{\ell/2} = \frac{4W}{\ell^4}\left(\frac{\ell^5}{64} - \frac{\ell^5}{160}\right)$$
$$= \underline{4W\ell}\times\frac{3}{320} = \frac{3}{80}W\ell$$

For the right hand half, from symmetry,

$C_{AB} = -\frac{3W\ell}{80}$; $C_{BA} = \frac{W\ell}{15}$

$\therefore$ For Case c, Fig. 10

$$C_{AB} = -\frac{W\ell}{15} - \frac{3W\ell}{80} = -\frac{25}{240}W\ell = -\frac{5}{48}W\ell$$

$$C_{BA} = +\frac{5}{48}W\ell.$$

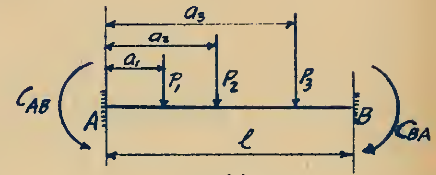


Fig. 11

Case d. Concentrated loads as shown in Fig. 11. For each load

$C_{AB} = -\frac{Pa(\ell-a)^2}{\ell^2}$ and $C_{BA} = +\frac{Pa^2(\ell-a)}{\ell^2}$.

For any number of concentrated loads

$$C_{AB} = -\frac{P_1a_1(\ell-a_1)^2}{\ell^2} - \frac{P_2a_2(\ell-a_2)^2}{\ell^2} - \frac{P_3a_3(\ell-a_3)^2}{\ell^2} - \dots\dots$$

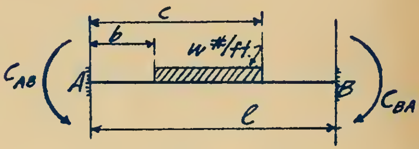


Fig. 12.

$$C_{BA} = +\frac{P_1a_1^2(\ell-a_1)}{\ell^2} + \frac{P_2a_2^2(\ell-a_2)}{\ell^2} + \frac{P_3a_3^2(\ell-a_3)}{\ell^2} + \dots\dots$$

Case e. Uniform load of w #/ft. as shown in Fig. 12. Load at any point distant x from A = w

$$d(C_{AB}) = -w\,dx.\frac{x(\ell-x)^2}{\ell^2}$$

$$C_{AB} = -\frac{w}{\ell^2}\int_b^c x(\ell-x)^2dx = -\frac{w}{\ell^2}\int_b^c(\ell^2x-2\ell x^2+x^3)dx = -\frac{w}{\ell^2}\left(\frac{\ell^2x^2}{2}-\frac{2\ell x^3}{3}+\frac{x^4}{4}\right)\Big/_b^c = -\frac{w}{\ell^2}\left(\frac{\ell^2c^2}{2}-\frac{2\ell c^3}{3}+\frac{c^4}{4}-\frac{\ell^2b^2}{2}+\frac{2\ell b^3}{3}-\frac{b^4}{4}\right)$$

$$= -\frac{W}{12\ell^2}\left[6\ell^2(c^2-b^2)+3(c^4-b^4)-8\ell(c^3-b^3)\right] = -\frac{W}{12\ell^2}\left[(\ell-b)^3(\ell+3b)-(\ell-c)^3(\ell+3c)\right]$$

$$d(C_{BA}) = w\,dx.x^2\frac{(\ell-x)}{\ell^2} \qquad C_{BA} = \frac{w}{\ell^2}\int_b^c(x^2\ell-x^3)dx$$

$$= \frac{w}{\ell^2}\left(\frac{\ell x^3}{3}-\frac{x^4}{4}\right)\Big/_b^c = \frac{W}{12\ell^2}\left[4\ell(c^3-b^3)-3(c^4-b^4)\right]$$

$$= \frac{W}{12\ell^2}\left[c^3(4\ell-3c)-b^3(4\ell-3b)\right]$$

∴ For Case e, Fig. 12

$$C_{AB} = -\frac{W}{12\ell^2}\left[(\ell-b)^3(\ell+3b) - (\ell-c)^3(\ell+3c)\right]$$

$$C_{BA} = +\frac{W}{12\ell^2}\left[c^3(4\ell-3c) - b^3(4\ell-3b)\right]$$

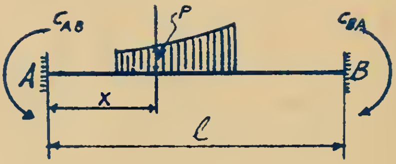


Fig. 13

General Case. See Fig. 13.

For continuous loads, where the loading curve is a function of x, as $p = f(x)$.

Then $d(C_{AB}) = -px\frac{(\ell-x)^2}{\ell^2}dx$

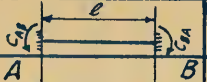
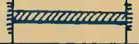
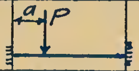
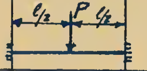
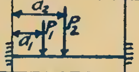
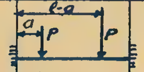
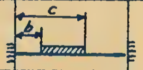
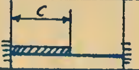
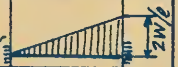
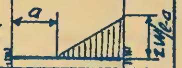
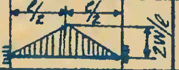
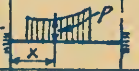
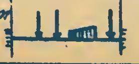
and $d(C_{BA}) = +px^2\frac{(\ell-x)}{\ell^2}dx.$

Hence $C_{AB} = -\int \frac{px(\ell-x)^2}{\ell^2}dx$

and $C_{BA} = \int \frac{px^2}{\ell^2}(\ell-x)\,dx$

For convenience the loading factors for the various conditions of loading discussed before are tabulated on page 28, Table I.

TABLE I. LOADING FACTORS.

Case	Loading		C_{AB}	C_{BA}	Remarks
	Loading				
1	w #/ft.		$-wl^2/12$	$+wl^2/12$	See page 17
2	Load P		$-\frac{Pa(l-a)^2}{l^2}$	$+\frac{Pa^2(l-a)}{l^2}$	" " 21
3	Load P		$-\frac{Pl}{8}$	$+\frac{Pl}{8}$	Special Case of #2
4	Loads P_1 & P_2		$-P_1a_1\frac{(l-a_1)^2}{l^2}-P_2a_2\frac{(l-a_2)^2}{l^2}$	$+P_1a_1^2\frac{(l-a_1)}{l^2}+P_2a_2^2\frac{(l-a_2)}{l^2}$	Special Case of #2
5	2 Loads P		$-\frac{Pa(l-a)}{l}$	$+\frac{Pa(l-a)}{l}$	Special Case of #2.
6	w #/ft.		$-\frac{w}{12l^2}[(l-b)^3(l+3b)-(l-c)^3(l+3c)]$	$+\frac{w}{12l^2}[c^3(4l-3c)-b^3(4l-3b)]$	See page 26
7	w #/ft.		$-\frac{w}{12l^2}[l^4-(l-c)^3(l+3c)]$	$+\frac{w}{12l^2}[c^3(4l-3c)]$	Special Case of #6
8	Total Load W		$-\frac{Wl}{15}$	$+\frac{Wl}{10}$	See page 24
9	Total Load W		$-\frac{W}{30l^2}(l-a)^2(2l+3a)$	$+\frac{W}{30l^2}(l-a)(3l^2+4la+3a^2)$	" " 24
10	Total Load W		$-\frac{5}{48}Wl$	$+\frac{5}{48}Wl$	" " 25
11	Any continuous load, $p=f(x)$		$-\int\frac{Px(l-x)^2}{l^2}dx$	$+\int\frac{Px^2(l-x)}{l^2}dx$	" " 27
12	Any system of loads		$\Sigma F(3\bar{x}-2l)/l^2$	$\Sigma F(3\bar{x}-l)/l^2$	" " 45
			where F is the area of the moment diagram for these loads on a simple beam and $\bar{x}$ is the distance from A of the center of gravity of that area.		

CHAPTER IV.

COMPARISON WITH OTHER METHODS.

In order to make a comparison between the various methods of analyzing statically indeterminate structures, it is necessary to start from a point common to all and proceed to the differences.

In examining the assumptions underlying the various methods it is found that they are the same in all of them. These assumptions are as follows: -

1. The deformations due to normal and shearing forces are relatively so small as to be negligible without appreciable error.
2. The joints are assumed to be rigid.
3. Horizontal and vertical members are assumed acting along single lines at or close to their neutral axes.
4. The length of a member is the length of the line representing it in the skeleton diagram between other lines intersecting it.
5. All formulas include the moment of inertia. Hence some preliminary assumptions as to what its value will be, have to be made.

From this common ground the differences between the various methods will now be examined. The following is an illustration of the analysis of an L Frame by the three methods.

A. L-FRAME BY THE SLOPE DEFLECTION METHOD

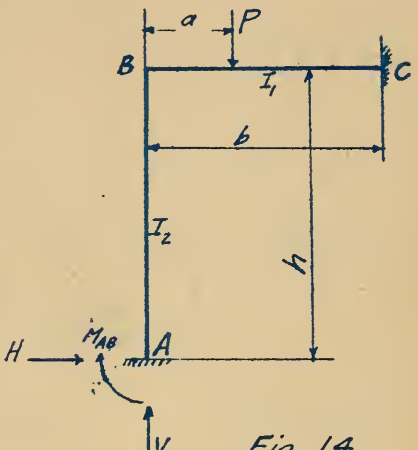


Fig. 14

Consider L frame shown in Fig. 14 loaded with any system of vertical loads. Let the moment of inertia of AB be I_2 and of BC, I_1.

B can have no movement relative to A and C, since any movement would mean a shortening of either AB, or BC, or both, which is against the assumption of no deformation due to normal forces (that is normal to the cross-section of the member).

In AB . $M_{AB} = \frac{2EI_2}{h}(2\theta_A + \theta_B - \frac{3d}{h})$ and $M_{BA} = \frac{2EI_2}{h}(\theta_A + 2\theta_B - \frac{3d}{h})$. Since d measures the movement of B from A, $d=0$ and

$$M_{AB} = \frac{2EI_2}{h}(2\theta_A + \theta_B)$$

$$M_{BA} = \frac{2EI_2}{h}(\theta_A + 2\theta_B)$$

In BC $M_{BC} = \frac{2EI_1}{b}(2\theta_B + \theta_C) + C_{BC}$ and $M_{CB} = \frac{2EI_1}{b}(\theta_B + 2\theta_C) + C_{CB}$

For equilibrium of joint B $\Sigma M = 0$, hence $M_{BA} + M_{BC} = 0$ or

$$\frac{2EI_2}{h}(\theta_A + 2\theta_B) + \frac{2EI_1}{b}(2\theta_B + \theta_C) + C_{BC} = 0 \text{ ———————— } (1)$$

Due to fixity of joints A and C, the tangents to the elastic curves at those joints equal zero. Hence

$\theta_A = 0$ and $\theta_C = 0$

Equation (1) becomes

$$4E\frac{I_2}{h}\theta_B + 4E\frac{I_1}{b}\theta_B + C_{BC} = 0 \text{ ——————— } (2)$$

$$4E\frac{\theta_B}{hb}(I_2 b + I_1 h) = -C_{BC}$$

$$\therefore \ \theta_B = \frac{-C_{BC} \cdot hb}{4E(I_2 b + I_1 h)}$$

Substitute in values for moments:

$$M_{AB} = \frac{2EI_2}{h}(2\theta_A + \theta_B) = 2\frac{EI_2}{h}\theta_B = -\frac{2EI_2}{h}\frac{hbC_{BC}}{4E(I_2b+I_1h)}) = -\frac{I_2 b}{2(I_2b+I_1h)}C_{BC}$$

$$M_{BA} = \frac{2EI_2}{h}(2\theta_B) = -\frac{4EI_2}{h}\ \frac{bh\ C_{BC}}{4E \times (I_2b+I_1h)} = -\frac{I_2 b}{(I_2b+I_1h)}C_{BC}$$

$$M_{BC} = \frac{2EI_1}{b}(2\theta_B) + C_{BC} = -\frac{4EI_1}{b}\ \frac{bh\ C_{BC}}{4E(I_2b+I_1h)} + C_{BC} = \frac{-I_1h\ C_{BC} + (I_2b+I_1h)C_{BC}}{I_2b+I_1h}$$

$$= +\frac{I_2 b}{I_2b+I_1h}C_{BC}$$

$$M_{CB} = \frac{2EI_1}{b}(\theta_B) + C_{CB} = -\frac{2EI_1}{b}\ \frac{bh\ C_{BC}}{(bI_2+I_1h)4E} + C_{CB} = -\frac{I_1 h}{(I_1h+I_2b)}\ \frac{C_{BC}}{2} + C_{CB}.$$

By considering AB a free member and applying principle $\Sigma M = 0$, H is obtained. Thus in Fig. 15
$\Sigma M = M_{AB} + M_{BA} - Hh = 0$ $\quad \therefore\ Hh = +M_{AB} + M_{BA} = -\frac{I_2 b}{(I_2b+I_1h)}\ \frac{3C_{BC}}{2}$

$$\therefore\ H = -\frac{I_2 b}{h(I_2b+I_1h)}\ \frac{3C_{BC}}{2}$$

By considering BC a free member and applying $\Sigma M = 0$, V is obtained. Thus in Fig. 16 $\Sigma M = M_{BC} + M_{CB} + Vb - \Sigma Pa_r = 0$ (Where a_r is lever arm of load).

$$\therefore\ Vb = \Sigma Pa_r - M_{BC} - M_{CB}$$

Results due to 3 cases of vertical loading are shown in Table III2. These results are obtained by substituting various values of C_{BC} and C_{CB} in the formulae

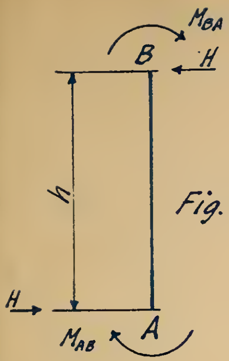


Fig. 15

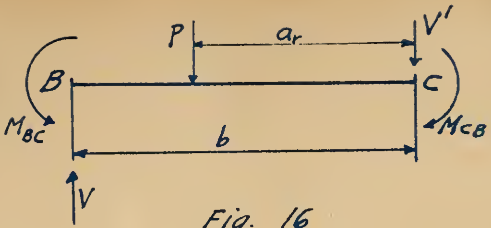
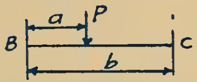
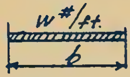
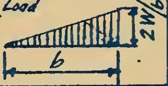


Fig. 16

TABLE 2

	Case 1	Case 2	Case 3
Load.			Total Load = W 
C_{BC}	$-Pa(b-a)^2/b^2$	$-wb^2/12$	$-Wb/15$
C_{BC}	$+Pa^2(b-a)/b^2$	$+wb^2/12$	$+Wb/10$
M_{AB}	$\frac{Pa(b-a)^2}{2b}\ \frac{I_2}{I_2b+I_1h}$	$\frac{wb^3}{24}\ \frac{I_2}{bI_2+hI_1}$	$\frac{Wb^2}{30}\ \frac{I_2}{I_2b+I_1h}$
M_{BA}	$\frac{Pa(b-a)^2}{b}\ \frac{I_2}{I_2b+I_1h}$	$\frac{wb^3}{12}\ \frac{I_2}{bI_2+hI_1}$	$\frac{Wb^2}{15}\ \frac{I_2}{I_2b+I_1h}$
M_{BC}	$-M_{BA}$	$-M_{BA}$	$-M_{BA}$
M_{CB}	$\frac{Pa(b-a)^2}{2b^2}\ \frac{I_1h}{(I_2b+I_1h)}+\frac{Pa^2(b-a)}{b^2}$	$\frac{wb^2}{24}\ \frac{I_1h}{(I_2b+I_1h)}+\frac{wb^2}{12}$	$\frac{Wb}{30}\ \frac{I_1h}{(I_2b+I_1h)}+\frac{Wb}{10}$
H	$\frac{3Pa(b-a)^2}{2bh}\ \frac{I_2}{I_2b+I_1h}$	$\frac{wb^3}{8h}\ \frac{I_2}{bI_2+hI_1}$	$\frac{Wb^2}{10h}\ \frac{I_2}{I_2b+I_1h}$
V	$\frac{Pa(b-a)^2(2bI_2-hI_1)}{2b^3(bI_2+hI_1)}+\frac{P(b-a)^2(b+a)}{b^3}$	$\frac{wb(3hI_1+4bI_2)}{8(hI_1+bI_2)}$	$\frac{W(2I_2b-I_1h)}{30(I_2b+I_1h)}+\frac{7W}{30}$

derived above.

B. L-FRAME BY THE METHOD OF LEAST WORK.

Consider L frame shown in Fig. 14.

Moment in member AB, $M = M_{AB} - Hx$

$$\frac{\partial M}{\partial M_{AB}} = 1 \; ; \quad \frac{\partial M}{\partial H} = -x \; ; \quad \frac{\partial M}{\partial V} = 0$$

$$\frac{\partial(\text{Work})}{\partial M_{AB}} = \frac{1}{I_2}\int_0^h (M_{AB} - Hx)\,dx = \frac{M_{AB}h}{I_2} - \frac{Hh^2}{2I_2} \quad (\text{Omitting } E).$$

$$\frac{\partial W}{\partial H} = \frac{1}{I_2}\int_0^h (M_{AB} - Hx)(-x)\,dx = \frac{Hh^3}{3I_2} - \frac{M_{AB}h^2}{2I_2}$$

$$\frac{\partial W}{\partial V} = 0$$

Moment in member BC $\quad M = M_{AB} - Hh + Vx$

$$\frac{\partial M}{\partial M_{AB}} = 1 \; ; \quad \frac{\partial M}{\partial H} = -h \; ; \quad \frac{\partial M}{\partial V} = +x$$

$$\frac{\partial W}{\partial M_{AB}} = \frac{1}{I_1}\int_0^b (M_{AB} - Hh + Vx)\,dx = M_{AB}\frac{b}{I_1} - \frac{Hhb}{I_1} + \frac{Vb^2}{2I_1}$$

$$\frac{\partial W}{\partial H} = -h \cdot \frac{\partial W}{\partial M_{AB}} = -M_{AB}\frac{bh}{I_1} + \frac{Hh^2b}{I_1} - \frac{Vb^2h}{2I_1}$$

$$\frac{\partial W}{\partial V} = \frac{1}{I_1}\int_0^b (M_{AB} - Hh + Vx)\,x\,dx = M_{AB}\frac{b^2}{2I_1} - \frac{Hhb^2}{2I_1} + \frac{Vb^3}{3I_1}$$

Additional moment in BC due to Load P (only to right of P), $M = -P(x-a)$

$\frac{\partial M}{\partial M_{AB}}$, $\frac{\partial M}{\partial H}$ and $\frac{\partial M}{\partial V}$ same as before for BC.

$$\frac{\partial W}{\partial M_{AB}} = -\frac{P}{I_1}\int_a^b (x-a)\,dx = -\frac{P}{I_1}\left(\frac{x^2}{2} - ax\right)\Big/_a^b = -\frac{P}{I_1}\frac{(b-a)^2}{2}$$

$$\frac{\partial W}{\partial H} = -h \times \frac{\partial W}{\partial M_{AB}} = \frac{Ph}{I_1}\frac{(b-a)^2}{2}$$

$$\frac{\partial W}{\partial V} = -\frac{P}{I_1}\int_a^b (x-a)\,x\,dx = -\frac{P}{I_1}\left(\frac{x^3}{3} - \frac{ax^2}{2}\right)\Big/_a^b = -\frac{P}{6I_1}(2b^3 - 3ab^2 + a^3)$$

Summation of work for both members due to each force, equals zero.

$$\Sigma \frac{\partial W}{\partial M_{AB}} = 0$$

$$\therefore \frac{M_{AB}h}{I_2} - \frac{Hh^2}{2I_2} + \frac{M_{AB}b}{I_1} - \frac{Hhb}{I_1} + \frac{Vb^2}{2I_1} - \frac{P(b-a)^2}{2I_1} = 0$$

$$\text{or } M_{AB}(hI_1 + bI_2) - \frac{Hh}{2}(hI_1 + 2bI_2) + \frac{Vb^2}{2}I_2 = \frac{PI_2}{2}(b-a)^2 \text{ ——— } (1).$$

$$\Sigma \frac{\partial W}{\delta H} = 0$$

$$\therefore \frac{Hh^3}{3I_2} - \frac{M_{AB}h^2}{2I_2} - \frac{M_{AB}bh}{I_1} + \frac{Hh^2b}{I_1} - \frac{Vb^2h}{2I_1} + \frac{Ph}{2I_1}(b-a)^2 = 0$$

$$\text{or } Hh(hI_1 + 3bI_2) - \frac{3}{2}M_{AB}(hI_1 + 2bI_2) - \frac{3}{2}Vb^2I_2 = -\frac{3}{2}PI_2(b-a)^2 \text{ —— } (2).$$

$$\Sigma \frac{\partial W}{\delta V} = 0$$

$$\therefore M_{AB}\frac{b^2}{2I_1} - \frac{Hhb^2}{2I_1} + \frac{Vb^3}{3I_1} - \frac{P}{6I_1}(2b^3 - 3ab^2 + a^3) = 0$$

$$\text{or } 3M_{AB}b^2 - 3Hhb^2 + 2Vb^3 = P(2b^3 - 3ab^2 + a^3) \text{ ——— } (3).$$

Eliminate V.

$$\text{From (1)} \quad Vb^2 = P(b-a)^2 - \frac{2M_{AB}}{I_2}(hI_1 + bI_2) + \frac{Hh}{I_2}(hI_1 + 2bI_2) \text{ —— } (4)$$

$$\text{From (2)} \quad Vb^2 = P(b-a)^2 - \frac{M_{AB}}{I_2}(hI_1 + 2bI_2) + \frac{2Hh}{3I_2}(hI_1 + 3bI_2) \text{ ——— } (5)$$

$$\text{From (3)} \quad Vb^2 = \frac{P}{2b}(2b^3 - 3ab^2 + a^3) - \frac{3M_{AB}b}{2} + \frac{3Hhb}{2} \text{ ——— } (6)$$

Subtract (4) from (5)

$$0 = \frac{M_{AB}}{I_2}(2hI_1 - hI_1) - \frac{Hh}{I_2}\left(hI_1 - \frac{2hI_1}{3}\right)$$

$$\text{or } \quad 0 = \frac{M_{AB}hI_1}{I_2} - \frac{Hh^2I_1}{3I_2}$$

$$\therefore \quad 3M_{AB} = Hh \text{ ——— } (7)$$

Subtract (6) from (4)

$$0 = P\left(-2ba + a^2 + \frac{3}{2}ab - \frac{a^3}{2b}\right) - \frac{M_{AB}}{I_2}\left(2hI_1 + 2bI_2 - \frac{3bI_2}{2}\right) + \frac{Hh}{I_2}\left(hI_1 + 2bI_2 - \frac{3bI_2}{2}\right)$$

or $$P\left(-\frac{ba}{2}+a^2-\frac{a^3}{2b}\right) = \frac{M_{AB}}{I_2}\left(2hI_1+\frac{bI_2}{2}\right) - \frac{Hh}{I_2}\left(hI_1+\frac{bI_2}{2}\right)$$

or $$-\frac{Pa}{2b}(b-a)^2 = \frac{M_{AB}}{2I_2}(4hI_1+bI_2) - \frac{Hh}{2I_2}(2hI_1+bI_2) \text{———} (8)$$

Eliminate H

From (8) $$Hh = \frac{Pa(b-a)^2}{b}\,\frac{I_2}{(2hI_1+bI_2)} + \frac{M_{AB}(4hI_1+bI_2)}{(2hI_1+bI_2)} \text{———} (9)$$

Equate (7) and (9)

$$3M_{AB} = \frac{Pa(b-a)^2 I_2}{b(2hI_1+bI_2)} + \frac{M_{AB}(4hI_1+bI_2)}{(2hI_1+bI_2)}$$

or $$M_{AB}\left[3-\frac{4hI_1+bI_2}{2hI_1+bI_2}\right] = \frac{Pa(b-a)^2 I_2}{b(2hI_1+bI_2)}$$

or $$M_{AB}(6hI_1+3bI_2-4hI_1-bI_2) = \frac{Pa}{b}(b-a)^2 I_2$$

or $$M_{AB}(2hI_1+2bI_2) = \frac{Pa}{b}(b-a)^2 I_2$$

or $$M_{AB} = \frac{Pa(b-a)^2}{2b}\,\frac{I_2}{(bI_2+hI_1)} \text{———} (10)$$

From (7) $$H = \frac{3M_{AB}}{h} = \frac{3Pa}{2bh}(b-a)^2\,\frac{I_2}{(bI_2+hI_1)} \text{———} (11)$$

Substitute (10) and (11) in (4)

$$Vb^2 = P(b-a)^2 - \frac{Pa(b-a)^2}{2b}\,\frac{I_2}{(bI_2+hI_1)}\,\frac{2(hI_1+bI_2)}{I_2} + \frac{3Pa(b-a)^2}{2bh}\,\frac{I_2}{(bI_2+hI_1)}\,\frac{h(hI_1+2bI_2)}{I_2}$$

or $$Vb^2 = P(b-a)^2 - \frac{\cancel{2}Pa(b-a)^2}{\cancel{2}b} + \frac{3Pa}{2b}(b-a)^2\,\frac{(hI_1+2bI_2)}{(bI_2+hI_1)}$$

or $$Vb^2 = \frac{P}{b}(b-a)^2\left[b-a+\frac{3a}{2}\,\frac{(hI_1+2bI_2)}{(bI_2+hI_1)}\right]$$

$$V = \frac{P(b-a)^3}{b^3} + \frac{3Pa(b-a)^2}{2b^3}\left(\frac{hI_1+2bI_2}{hI_1+bI_2}\right) \text{———} (12)$$

This value of V may be changed to the form given in Case 1, Table 2.

Equations (10), (11) and (12) give values of M_{AB}, H and V for a load P on BC distant a from B. Other cases of vertical loading are not worked out here.

C. L-FRAME BY THE ELASTIC METHOD

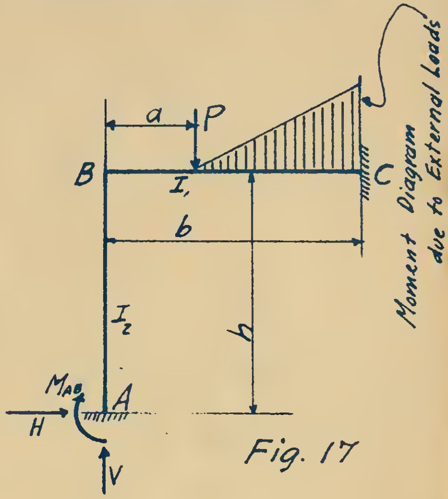


Fig. 17

Assume the frame cut at A (see Fig. 17). M_{AB}, H and V, the forces at this section are the unknowns. Assume origin of coördinates at A.

Since A is a fixed joint, there will be no deflection of end A when the frame is cut. The tangent to the elastic curve will be equal zero at A.

The fundamental equations are

$$\int \frac{M_e y\,ds}{EI} + \int \frac{M_r y\,ds}{EI} = 0$$

$$\int \frac{M_e x\,ds}{EI} - \int \frac{M_r x\,ds}{EI} = 0$$

$$\int \frac{M_e\,ds}{EI} + \int \frac{M_r\,ds}{EI} = 0 \quad *$$

---

* See "Modern Framed Structures" Vol. 2, by Johnson, Bryan and Turneaure, p. 171 (1917).

Due to the fixity of joint A these equations reduce to

$$\int M_r y \frac{ds}{EI} = 0 \quad ; \quad \int M_r x \frac{ds}{EI} = 0 \quad ; \quad \int M_r \frac{ds}{EI} = 0$$

Cancel E (as a constant) and drop the subscripts of M.

$$\therefore \int M y \frac{ds}{I} = 0 , \quad \int M x \frac{ds}{I} = 0 \quad \text{and} \quad \int M \frac{ds}{I} = 0$$

where M is the moment at any point.

In Fig. 17 it is seen that

$$M = M' + M_{AB} - H\,y + V\,x$$

where M' is the moment due to external loads. Substitute this value of M in the fundamental equations. Then

$$\int M' x \frac{ds}{I} + M_{AB} \int \frac{x\,ds}{I} - H \int xy \frac{ds}{I} + V \int x^2 \frac{ds}{I} = 0 \text{ ——— } (1)$$

$$\int M' y \frac{ds}{I} + M_{AB} \int \frac{y\,ds}{I} - H \int y^2 \frac{ds}{I} + V \int xy \frac{ds}{I} = 0 \text{ ——— } (2)$$

$$\int M' \frac{ds}{I} + M_{AB} \int \frac{ds}{I} - H \int y \frac{ds}{I} + V \int \frac{x\,ds}{I} = 0 \text{ ——— } (3)$$

The values of the integrals in the equations (1), (2) and (3) for the dimensions of the frame and for the load are shown in Table 3.

Note that the origin is at A; y is positive when measured upwards; x is positive when measured to the right and M' is the moment caused by P.

Substitute the values of the integrals in equations (1), (2) and (3).

TABLE 3.

Integral	A to B	B to C	Integral	A to B	B to Load	Load to C
$\int \frac{ds}{I}$	h/I_2	b/I_1	M'	0	0	$-P(x-a)$
$\int x\frac{ds}{I}$	0	$b^2/2I_1$	$\int M' ds/I$	0	0	$-P\frac{(b-a)^2}{2I_1}$
$\int y\frac{ds}{I}$	$h^2/2I_2$	hb/I_1	$\int M' x\, ds/I$	0	0	$-\frac{P}{6I_1}(2b^3-3ab^2+a^3)$
$\int xy\frac{ds}{I}$	0	$hb^2/2I_1$	$\int M' y\, ds/I$	0	0	$-\frac{Ph}{2I_1}(b-a)^2$
$\int x^2\frac{ds}{I}$	0	$b^3/3I_1$				
$\int y^2\frac{ds}{I}$	$h^3/3I_2$	h^2b/I_1				

Equation (1) becomes

$$-\frac{P}{6I_1}(2b^3-3ab+a^3)+M_{AB}\frac{b^2}{2I_1}-H\frac{hb^2}{2I_1}+V\frac{b^3}{3I_1}=0$$

or $$3M_{AB}b^2-3Hhb^2+2Vb^3 = P(2b^3-3ab^2+a^3) \text{———} (4)$$

Equation (2) becomes

$$-\frac{Ph}{2I_1}(b-a)^2+M_{AB}h\left(\frac{h}{2I_2}+\frac{b}{I_1}\right)-Hh^2\left(\frac{h}{3I_2}+\frac{b}{I_1}\right)+V\frac{hb^2}{2I_1}=0$$

or $$\frac{3}{2}M_{AB}(hI_1+2bI_2)-Hh(hI_1+3bI_2)+\frac{3}{2}Vb^2I_2 = \frac{3}{2}PI_2(b-a)^2 \text{———} (5)$$

Equation (3) becomes

$$-\frac{P}{2I_1}(b-a)^2+M_{AB}\left(\frac{h}{I_2}+\frac{b}{I_1}\right)-Hh\left(\frac{h}{2I_2}+\frac{b}{I_1}\right)+V\frac{b^2}{2I_1}=0$$

or $$M_{AB}(hI_1+bI_2)-\frac{Hh}{2}(hI_1+2bI_2)+\frac{Vb^2I_2}{2} = \frac{PI_2(b-a)^2}{2} \text{———} (6)$$

Equations (4), (5) and (6) are the same as equations (3), (2) and (1) respectively (page 34) worked out by the method of least work.

The solution of the equations (4), (5) and (6) is not shown here. The procedure is identical with the one followed for the method of least work.

Values of M_{AB}, H and V are therefore the same as for least work.

Other cases of vertical loading are not worked

out here.

DISCUSSION OF ADVANTAGES.

1. The fundamental equation of the slope deflection method

$$M_{AB} = \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3d}{l}\right) + C_{AB}$$

is simpler and easier to remember than the equations of the elastic method

$$\int \frac{M_e y\,ds}{EI} + \int \frac{M_r y\,ds}{EI} = 0 \;; \quad \int \frac{M_e x\,ds}{EI} - \int \frac{M_r x\,ds}{EI} = 0 \;;$$

$$\int \frac{M_e\,ds}{EI} + \int \frac{M_r\,ds}{EI} = 0$$

The equation of the least work method

$$\int \frac{M}{EI}\,\frac{\partial M}{\partial F_x}\,dx = 0$$

where F_x is any unknown force or moment, is also simpler than the equations of the elastic method.

2. The fundamental equation of the slope deflection method is easy to apply to a case in hand, involving a knowledge of only the loading factor and the moment of inertia. The loading factor for all usual conditions of load is given in Table 1, Chapter III (as well as in several other publications).

An application of the fundamental equations of the elastic and least work methods to a problem involves the moment of inertia and, most important, several integrations.

3. The slope deflection method does not involve the use of calculus (for ordinary conditions of loading); while the other methods depend upon calculus in solving for any condition of loading.

4. The combining of the fundamental equations of the slope deflection method into the general equations for the frame (equation 2 in the example) is a matter of addition of algebraic terms;

whereas in the other methods (see example for obtaining equations 1, 2, and 3 by least work and equations 4, 5, and 6 by the elastic) it is a complicated integrating process.

5. The frame equations thus obtained by the slope deflection method (equation 2) are in a form which lends itself to a quicker solution, than that of the equations obtained by least work (equation 1, 2, and 3) and the elastic (equation 4, 5, and 6) methods.

6. By means of the slope deflection method moments at the joints can be worked out for a general case of loading; specific cases may be substituted afterwards without many computations (see Table 2).

In the other methods involving integrations of terms containing the load, such a procedure would complicate the frame equations to an extent where the usefulness of this step would be lost.

To resume; the example worked out above is an illustration of the ease of applying, the fewer computations and absence of integrations in the slope deflection method.

In more complicated frames the analysis by means of the elastic and least work methods becomes still more complicated as compared with the slope deflection method.

Chapter V.

APPLICATION TO BEAMS WITH VARYING MOMENTS OF INERTIA

A. Derivation of the Fundamental Equations.

Let AB (Fig. 18) be part of a frame; M_{AB} and M_{BA} are moments caused by the restraint of the frame. AB is the original position of the member; A'B is the neutral axis after deformation. A' is deflected a distance d from its original position.

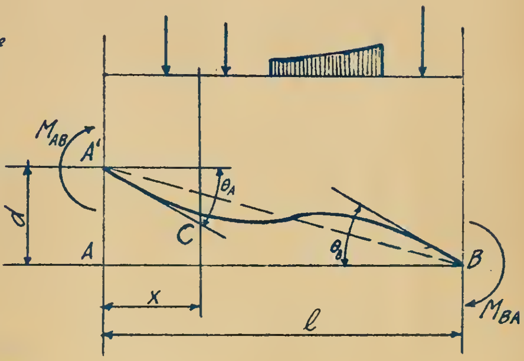


Fig. 18

Assume any system of loads on the member.

Signs of quantities are the same as in chapter III.

The equation of the elastic curve is

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} \quad *$$

For member AB, E is assumed constant and I is assumed varying from point to point.

---

* See A. H. Heller, Stresses in Structures, p. 128.

Hence $E\frac{d^2y}{dx^2} = -\frac{M}{I}$

The moment at any point C due to restraint,

$M_{CB} = M_{AB}\left(\frac{l-x}{l}\right) - M_{BA}\frac{x}{l}$ (See page 14)

Let the moment due to the loading acting as on a simple beam be $M_{CB} = Pf(x^n)$ *, where f means a function of.

The total moment at C is

$$M_{CB} = M_{AB}\left(\frac{l-x}{l}\right) - M_{BA}\frac{x}{l} + Pf(x^n).$$

Substitute this value of M_{CB} in the general equation:

$$E\frac{d^2y}{dx^2} = -\frac{M_{AB}}{I} + \frac{M_{AB}}{l}\frac{x}{I} + \frac{M_{BA}}{l}\frac{x}{I} - Pf\left(\frac{x^n}{I}\right). \text{——————} (1)$$

Integrate

$$E\frac{dy}{dx} = -M_{AB}\int\frac{dx}{I} + \frac{M_{AB}}{l}\int\frac{x\,dx}{I} + \frac{M_{BA}}{l}\int\frac{x\,dx}{I} - P\int f\left(\frac{x^n}{I}\right)dx + C_1 \text{——} (2)$$

When $x=0$, $\frac{dy}{dx} = \theta_A$.

$$\therefore \quad E\theta_A = C_1$$

When $x = l$ $\frac{dy}{dx} = \theta_B$.

Note also that as $Pf(x^n)$ is the moment due to the loading, $Pf\left(\frac{x^n}{I}\right)$ is an ordinate in the $\frac{M}{I}$ diagram due to that loading and $P\int f\left(\frac{x^n}{I}\right)dx$ is the area of that $\frac{M}{I}$ diagram. Call this area $\frac{F}{I}$.

Substitute in equation (2) for $x = l$

$$E\theta_B = -M_{AB}\int\frac{dx}{I} + \frac{M_{AB}}{l}\int\frac{x\,dx}{I} + \frac{M_{BA}}{l}\int\frac{x\,dx}{I} - \frac{F}{I} + E\theta_A$$

---

* The expression for moment due to a concentrated load on a simple beam is not the same thruout the length of the beam. Thus, in Fig. 18a, the moment to the left of the load is $M = Rx$ and to the right of the load is $M = Rx - P(x-a)$. $Pf(x^n)$ is the general form of expression for moment due to any load.

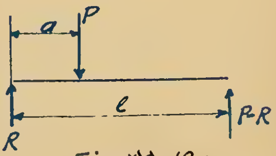


Fig. 18a

or $$E(\theta_B - \theta_A) = -M_{AB}\int\frac{dx}{I} + \frac{M_{AB}}{l}\int\frac{x\,dx}{I} + \frac{M_{BA}}{l}\int\frac{x\,dx}{I} - \frac{F}{I}$$

or $$E(\theta_B l - \theta_A l) = -M_{AB}\,l\int\frac{dx}{I} + M_{AB}\int\frac{x\,dx}{I} + M_{BA}\int\frac{x\,dx}{I} - \frac{F}{I}.l \text{ ——————— } (3)$$

Integrate (2), remembering that $C_1 = E\theta_A$

$$Ey = -M_{AB}\iint\frac{dx}{I}dx + \frac{M_{AB}}{l}\iint\frac{x\,dx}{I}dx + \frac{M_{BA}}{l}\iint\frac{x\,dx}{I}dx - P\iint f\left(\frac{x^n}{I}\right)dx\,dx + E\theta_A x + C_2$$

When $x=0$ $y=-d$ $\therefore$ $C_2 = -Ed$

~~When $x=l$ $y=0$~~

$$\therefore Ey = -M_{AB}\iint\frac{dx}{I}dx + \frac{M_{AB}}{l}\iint\frac{x\,dx}{I}dx + \frac{M_{BA}}{l}\iint\frac{x\,dx}{I}dx - P\iint f\left(\frac{x^n}{I}\right)dx\,dx + E\theta_A x - Ed$$

It can be proved that

$\iint\frac{dx}{I}dx = x\int\frac{dx}{I} - \int\frac{x\,dx}{I}$, where I is a variable

$\iint\frac{x\,dx}{I}dx = x\int\frac{x\,dx}{I} - \int\frac{x^2dx}{I}$, " " " " "

and in general

$\iint f\left(\frac{x^n}{I}\right)dx\,dx = x\int f\left(\frac{x^n}{I}\right)dx - \int f\left(\frac{x^{n+1}}{I}\right)dx$, " " " "

$$\therefore Ey = -M_{AB}\left[x\int\frac{dx}{I} - \int\frac{x\,dx}{I}\right] + \frac{M_{AB}}{l}\left[x\int\frac{x\,dx}{I} - \int\frac{x^2dx}{I}\right] + \frac{M_{BA}}{l}\left[x\int\frac{x\,dx}{I} - \int\frac{x^2dx}{I}\right]$$
$$- Px\int f\left(\frac{x^n}{I}\right)dx + P\int f\left(\frac{x^{n+1}}{I}\right)dx + E\theta_A x - Ed \text{ ——————— } (4)$$

But $P\int f\left(\frac{x^n}{I}\right)dx$ is the area of the $\frac{M}{I}$ diagram for the loading and $P\int f\left(\frac{x^{n+1}}{I}\right)dx$ is the moment of that area about A. Let $\frac{F}{I}$ be the area of $\frac{M}{I}$ diagram and $\bar{x}$ the distance of its center of gravity from A. Then

$P\int f\left(\frac{x^n}{I}\right)dx = \frac{F}{I}$ and $P\int f\left(\frac{x^{n+1}}{I}\right)dx = \frac{F}{I}.\bar{x}$

when $x = l$, $y=0$

Substitute these values in (4)

$$0 = -M_{AB}\left[l\int\frac{dx}{I} - \int\frac{x\,dx}{I}\right] + M_{AB}\left[\int\frac{x\,dx}{I} - \frac{1}{l}\int\frac{x^2dx}{I}\right] + M_{BA}\left[\int\frac{x\,dx}{I} - \frac{1}{l}\int\frac{x^2dx}{I}\right] - l\frac{F}{I}$$
$$+ \frac{F}{I}\bar{x} + E\theta_A l - Ed \text{ ——————— } (5)$$

or $$E(d - \theta_A l) = M_{AB}\left[2\int\frac{x\,dx}{I} - l\int\frac{dx}{I} - \frac{1}{l}\int\frac{x^2dx}{I}\right] + M_{BA}\left[\int\frac{x\,dx}{I} - \frac{1}{l}\int\frac{x^2dx}{I}\right] - l\frac{F}{I} + \frac{F}{I}\bar{x} \quad (6)$$

Eliminate M_{BA}

From (3) $$E(\theta_B l-\theta_A l)\left(\int\frac{x\,dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right) = M_{AB}\left(\int\frac{x\,dx}{I}-l\int\frac{dx}{I}\right)\left(\int\frac{x\,dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right)$$
$$+M_{BA}\int\frac{x\,dx}{I}\left(\int\frac{x\,dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right)-\frac{F}{I}\cdot l\left(\int\frac{x\,dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right) \text{———} \quad (3a)$$

From (6) $$E(d-\theta_A l)\int\frac{x\,dx}{I} = M_{AB}\left[2\int\frac{x\,dx}{I}-l\int\frac{dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right]\int\frac{x\,dx}{I}$$
$$+M_{BA}\left[\int\frac{x\,dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right]\int\frac{x\,dx}{I}+\left(\frac{F}{I}\bar{x}-\frac{lF}{I}\right)\left(\int\frac{x\,dx}{I}\right) \text{———} \quad (6a)$$

Subtract (6a) from (3a)

$$E\left(\theta_B l\int\frac{x\,dx}{I}-\theta_B\int\frac{x^2dx}{I}-\cancel{\theta_A l\int\frac{x\,dx}{I}}+\theta_A\int\frac{x^2dx}{I}-d\int\frac{x\,dx}{I}+\cancel{\theta_A l\int\frac{x\,dx}{I}}\right) = M_{AB}\left(\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}\right.$$
$$-\frac{1}{l}\cancel{\int\frac{x\,dx}{I}}\int\frac{x^2dx}{I}-l\cancel{\int\frac{dx}{I}}\int\frac{x\,dx}{I}+\int\frac{dx}{I}\int\frac{x^2dx}{I}-2\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}+l\cancel{\int\frac{dx}{I}}\int\frac{x\,dx}{I}+\frac{1}{l}\cancel{\int\frac{x\,dx}{I}}\left.\int\frac{x^2dx}{I}\right)$$
$$+\frac{F}{I}\left(\int\frac{x^2dx}{I}-\cancel{l\int\frac{x\,dx}{I}}-\bar{x}\int\frac{x\,dx}{I}+\cancel{l\int\frac{x\,dx}{I}}\right)$$

or $$E\left[\theta_A\int\frac{x^2dx}{I}+\theta_B\left(l\int\frac{x\,dx}{I}-\int\frac{x^2dx}{I}\right)-d\int\frac{x\,dx}{I}\right] = M_{AB}\left(\int\frac{dx}{I}\int\frac{x^2dx}{I}-\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}\right)$$
$$+\frac{F}{I}\left(\int\frac{x^2dx}{I}-\bar{x}\int\frac{x\,dx}{I}\right)$$

$$\therefore M_{AB} = \frac{E\left[\theta_A\int\frac{x^2dx}{I}+\theta_B\left(l\int\frac{x\,dx}{I}-\int\frac{x^2dx}{I}\right)-d\int\frac{x\,dx}{I}\right]-\frac{F}{I}\left(\int\frac{x^2dx}{I}-\bar{x}\int\frac{x\,dx}{I}\right)}{\left(\int\frac{dx}{I}\int\frac{x^2dx}{I}-\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}\right)} \quad (7)$$

Eliminate M_{AB}

From (3) $$E(\theta_B l-\theta_A l)\left(2\int\frac{x\,dx}{I}-l\int\frac{dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right) = M_{AB}\left(\int\frac{x\,dx}{I}-l\int\frac{dx}{I}\right)\left(2\int\frac{x\,dx}{I}\right.$$
$$\left.-l\int\frac{dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right)+M_{BA}\int\frac{x\,dx}{I}\left(2\int\frac{x\,dx}{I}-l\int\frac{dx}{I}-\frac{1}{l}\int x^2dx\right)-\frac{F}{I}l\left(2\int\frac{x\,dx}{I}-l\int\frac{dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right) \quad (3b)$$

From (6) $$E(d-\theta_A l)\left(\int\frac{x\,dx}{I}-l\int\frac{dx}{I}\right) = M_{AB}\left(2\int\frac{x\,dx}{I}-l\int\frac{dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right)\left(\int\frac{x\,dx}{I}-l\int\frac{dx}{I}\right)$$
$$+M_{BA}\left(\int\frac{x\,dx}{I}-\frac{1}{l}\int\frac{x^2dx}{I}\right)\left(\int\frac{x\,dx}{I}-l\int\frac{dx}{I}\right)-\frac{F}{I}(l-\bar{x})\left(\int\frac{x\,dx}{I}-l\int\frac{dx}{I}\right) \text{———} \quad (6b)$$

Subtract (6b) from (3b)

$$E\left(2\theta_B l\int\frac{x\,dx}{I}-\theta_B l^2\int\frac{dx}{I}-\theta_B\int\frac{x^2dx}{I}-2\theta_A l\int\frac{x\,dx}{I}+\cancel{\theta_A l^2\int\frac{dx}{I}}+\theta_A\int\frac{x^2dx}{I}\right)-E\left(d\int\frac{x\,dx}{I}-dl\int\frac{dx}{I}\right.$$
$$\left.-\theta_A l\int\frac{x\,dx}{I}+\cancel{\theta_A l^2\int\frac{dx}{I}}\right) = M_{BA}\left(2\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}-l\cancel{\int\frac{dx}{I}}\int\frac{x\,dx}{I}-\frac{1}{l}\cancel{\int\frac{x\,dx}{I}}\int\frac{x^2dx}{I}-\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}\right.$$
$$\left.+l\cancel{\int\frac{dx}{I}}\int\frac{x\,dx}{I}+\frac{1}{l}\cancel{\int\frac{x\,dx}{I}}\int\frac{x^2dx}{I}-\int\frac{dx}{I}\int\frac{x^2dx}{I}\right)-\frac{F}{I}\left(l\int\frac{x\,dx}{I}-\cancel{l^2\int\frac{dx}{I}}-\int\frac{x^2dx}{I}-l\int\frac{x\,dx}{I}+\cancel{l^2\int\frac{dx}{I}}\right.$$
$$\left.+\bar{x}\int\frac{x\,dx}{I}-\bar{x}l\int\frac{dx}{I}\right)$$

or $$E\left[-\theta_A \ell\int\frac{x\,dx}{I}+\theta_A\int\frac{x^2dx}{I}+\theta_B\left(2\ell\int\frac{x\,dx}{I}-\ell^2\int\frac{dx}{I}-\int\frac{x^2dx}{I}\right)-d\left(\int\frac{x\,dx}{I}-\ell\int\frac{dx}{I}\right)\right]$$

$$=M_{BA}\left(\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}-\int\frac{dx}{I}\int\frac{x^2dx}{I}\right)-\frac{F}{I}\left(\ell\int\frac{x\,dx}{I}-\int\frac{x^2dx}{I}+\bar{x}\int\frac{x\,dx}{I}-\bar{x}\,\ell\int\frac{dx}{I}\right)$$

$$\therefore\; M_{BA}=\frac{E\left[\theta_A\left(\ell\int\frac{x\,dx}{I}-\int\frac{x^2dx}{I}\right)+\theta_B\left(\ell^2\int\frac{dx}{I}-2\ell\int\frac{x\,dx}{I}+\int\frac{x^2dx}{I}\right)-d\left(\ell\int\frac{dx}{1}-\int\frac{x\,dx}{I}\right)\right]}{\left(\int\frac{dx}{I}\int\frac{x^2dx}{I}-\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}\right)}$$

$$+\frac{\frac{F}{I}\left(\bar{x}\,\ell\int\frac{dx}{I}-\bar{x}\int\frac{x\,dx}{I}-\ell\int\frac{x\,dx}{I}+\int\frac{x^2dx}{I}\right)}{\left(\int\frac{dx}{I}\int\frac{x^2dx}{I}-\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}\right)} \tag{8}$$

Equations 7 and 8 give values of M_{AB} and M_{BA} when I is variable. For any practical case the member AB may be divided into any number of parts Δx, the moment of inertia for each part found and substituted into equations 7 and 8, changing the integrals $\int$ to summations Σ.

Equations 7 and 8 contain general expressions for loading factors due to any condition of load. It may be useful to find the corresponding general expressions for loading factors for a member of constant moment of inertia.

$$C_{AB}=-\frac{F}{I}\left(\int\frac{x^2dx}{I}-\bar{x}\int\frac{x\,dx}{I}\right)\Big/\left(\int\frac{dx}{I}\int\frac{x^2dx}{I}-\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}\right)$$

$$C_{BA}=+\frac{F}{I}\left(\bar{x}\,\ell\int\frac{dx}{I}-\bar{x}\int\frac{x\,dx}{I}-\ell\int\frac{x\,dx}{I}+\int\frac{x^2dx}{I}\right)\Big/\left(\int\frac{dx}{I}\int\frac{x^2dx}{I}-\int\frac{x\,dx}{I}\int\frac{x\,dx}{I}\right)$$

Make I constant and integrate with limits from 0 to ℓ.

$$C_{AB}=-\frac{F}{I^2}\left(\frac{\ell^3}{3}-\bar{x}\frac{\ell^2}{2}\right)\Big/\frac{1}{I^2}\left(\ell\frac{\ell^3}{3}-\frac{\ell^2}{2}\frac{\ell^2}{2}\right)$$

$$=-\frac{F\left(\frac{\ell}{3}-\frac{\bar{x}}{2}\right)}{\frac{\ell^2}{3}-\frac{\ell^2}{4}}=-\frac{F(2\ell-3\bar{x})12}{6\,\ell^2}=-\frac{2F(2\ell-3\bar{x})}{\ell^2}$$

$$C_{BA}=+\frac{F}{I^2}\left(\bar{x}\,\ell^2-\bar{x}\frac{\ell^2}{2}-\frac{\ell^3}{2}+\frac{\ell^3}{3}\right)\Big/\frac{1}{I^2}\left(\ell\frac{\ell^3}{3}-\frac{\ell^2}{2}\frac{\ell^2}{2}\right)$$

$$= \frac{F\left(\frac{\bar{x}}{2}-\frac{\ell}{6}\right)}{\left(\ell^2/3-\ell^2/4\right)} = \frac{F(3\bar{x}-\ell)12}{6\,\ell^2} = \frac{2F(3\bar{x}-\ell)}{\ell^2}.$$

Where F is the area of the moment diagram of the load on a simple beam
and $\bar{x}$ is the distance from A of the center of gravity of that area.

These values of C_{AB} and C_{BA} are placed in Table I, page 28.

B. APPLICATION OF FUNDAMENTAL EQUATIONS.

Due to various factors * the moment of inertia of concrete beams (and sometimes of steel beams) is not constant throughout their entire length. In concrete beams the moment of inertia varies practically from point to point. The equations derived above have a special application in a number of cases of beams restrained at the ends.

1. In bridge and in building construction one encounters sometimes girders carrying unusually heavy loads, but limited in size by some considerations. In such girders, where in order to increase the resisting moment, advantage is taken of a haunch near the support, or of a curved underside, or of T-action, the moment of inertia will vary considerably and the variation will have an effect on the distribution of the bending moments. In beams of smaller size it is easy to provide extra material and so to take care of an unknown distribution of bending moments. But in heavy girders such practice would be uneconomical and unsafe. A more exact knowledge of the distribution of bending moments may be obtained by the use of the above equations.

2. In the usual designing methods a constant moment of inertia of concrete beams is assumed. For designing ordinary beams this is close enough, but it is necessary to know the limitations of the designing methods. In most concrete beams there is a haunch near the support, increasing the negative moment at the support beyond the value usually assigned to it, and diminishing the positive moment at the center.

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* These factors are discussed in the next chapter.

Sometimes there are T-flanges in the middle portion of the beam. This changes the moment of inertia of the beam and hence affects the distribution of bending moments. The extent of the influence of these factors can be readily determined by the use of the equations derived above.

3. Take a beam and continually decrease its center section. Each end of the beam will then approach in action a cantilever — until, when the center section becomes zero, it will become a cantilever. The moment at the center will vary from $\frac{wl^2}{24}$ (considering beams with restrained ends only) to zero. For any intermediate case the moment at the center can be obtained by using the equations derived above. This is a useful application in cases where for practical considerations the center section cannot be made heavy enough to resist to bending moment coming on it. The beam may then be designed with a distribution of the moment of inertia along its axis, which will increase the bending moment at the supports and decrease it at the center until it becomes equal to the resisting moment of that section.

4. By introducing a factor containing the effect of the horizontal thrust, the above equations will be modified to apply to arches. By this method continuous arches of several spans may be analyzed.

CHAPTER VI.

DETAILS.

The fundamental equations of the Slope Deflection method show that the moment at the joint of a member depends on the moment of inertia of the member. Thus in the equation

$$M_{AB} = \frac{2EI}{\ell}\left(2\theta_A + \theta_B - \frac{3d}{\ell}\right) + C_{AB},$$

the term I stands for the moment of inertia of the member.

In analyzing rigid frames the fundamental equations for the ends of each member are written out. The equations for the members around each joint are summed up and equated to zero. By solving these equations the moments at the joints are obtained.

Let n members meet at any joint O of a frame. The fundamental equations for the end O of the members meeting at that joint will be of the type:

$$M_{OC} = \frac{2EI_{OC}}{\ell_{OC}}\left(2\theta_O + \theta_C - \frac{3d_{OC}}{\ell_{OC}}\right) + C_{OC}$$

The sum of these equations around joint 0 equals zero. Hence

$$\frac{2EI_{OA}}{\ell_{OA}}\left(2\theta_O + \theta_A - \frac{3d_{OA}}{\ell_{OA}}\right) + C_{OA} \quad + \frac{2EI_{OB}}{\ell_{OB}}\left(2\theta_O + \theta_B - \frac{3d_{OB}}{\ell_{OB}}\right) + C_{OB}$$

$$+ \ldots\ldots \quad + \frac{2EI_{ON}}{\ell_{ON}}\left(2\theta_O + \theta_N - \frac{3d_{ON}}{\ell_{ON}}\right) + C_{ON} = 0 \quad \text{------} (1)$$

Divide thru by I_{OA},

Equation (1) becomes:

$$\frac{2E}{\ell_{OA}}\left(2\theta_O + \theta_A - \frac{3d_{OA}}{\ell_{OA}}\right) + \frac{C_{OA}}{I_{OA}} \quad + \frac{2EI_{OB}}{\ell_{OB}I_{OA}}\left(2\theta_O + \theta_B - \frac{3d_{OB}}{\ell_{OB}}\right) + \frac{C_{OB}}{I_{OA}}$$

$$+ \ldots\ldots + \frac{2EI_{ON}}{\ell_{ON}I_{OA}}\left(2\theta_O + \theta_N - \frac{3d_{ON}}{\ell_{ON}}\right) + \frac{C_{ON}}{I_{OA}} = 0 \quad \text{------} (2)$$

When the equations for all joints have been solved simultaneously, the unknowns will be found to have values of the type:

$$\frac{I}{I_{OA}} \cdot f\left(C_{OA}, C_{OB}, \ldots\ldots C_{ON}\right)$$

(where f means a function of).

Substituting these values of the unknowns into the fundamental equations for moments, will give values of the moments at the joints.

All these values will then contain a factor of the type $\frac{I}{I_{oA}}$. Thus the moments at the joints depend upon the ratio of the moments of inertia of the various members of the frame. From this follows, that when the ratio of the moments of inertia is known, the moments at the joints may be obtained, even though the actual values of the moments of inertia are not known.

This is of great value, as in concrete construction only relative values of the moment of inertia may be calculated.

The following is a discussion of the methods for obtaining approximate values of I[*] for members of concrete frames, and their influence on the relative values of I for various members of the frame.

In general members of concrete frames are divided into two classes:

1. Beams, or members subjected mainly to bending stresses, and
2. Columns, or members subjected mainly to direct stresses.

To obtain the I of a concrete beam is a complicated problem. The ultimate tensile strength of concrete is about 250 or 300 lbs. per sq. in. When the beam is loaded so that the reinforcing steel is stressed to 16000-20000 lbs. per sq. in., the tension side of the beam cracks up to about the neutral axis.

[*] I stands for the moment of inertia.

In this case the full section is not effective in obtaining I of the beam. The area effective for I is the part of the section extending from the compressive face of the section to some distance below the neutral axis. This distance can only approximately be ascertained by analytical methods. It depends to a great extent on the qualities of the concrete at and near the section considered.

Another difficulty encountered in getting I of concrete beams is the fact that it is not uniform throughout the length of the beam, but varies from section to section. In a simple beam of constant cross-section this variation is not great, since it is due only to the bent up rods and to the stirrups. But in fixed or restrained beams, where at the ends, there are bending moments of some magnitude, and the beam is changed in cross section, this variation of I becomes more important.

The case of columns is different. The compression due to the direct load balances in most cases the tension due to bending. Even when the tension exceeds the compression by 100 or 150 lbs. per sq. in., there are still no cracks in the section. This allows the use of the full section in finding I of a column.

The moment at the joint of a column and a beam depends on the ratio of I's of the beam and column. When the I's of the beams and columns are obtained with the same degree of accuracy the resulting value of the moment at the joint will be correct. But when I of one member is found more accurately than of the other, the ratio of I's and hence the value of the moment at the joint will not be correct.

While it is easy to obtain a more or less accurate value of I of a column, it is difficult to obtain I of a beam. It then becomes important to devise some methods of finding the I of a beam with about the same degree of accuracy as that of a column.

A. I OF A SECTION AT MAXIMUM MOMENT.

As was mentioned before the concrete on the tension side of a loaded beam cracks and leaves only a part of the section effective for I. Since concrete can take a small amount of tension, the cracks will not go to the neutral axis. This is the basis on which Mr. S. E. Thompson calculated I for beams when making studies for his "Treatise on Concrete".*

Prof. R. E. Spaulding* agrees substantially with such a procedure, although he thinks that the part of the section taking tension is close to the neutral axis and has only a slight influence on I. Accordingly he uses for obtaining I only that part of the concrete which is in compression.

To show the variation in the value of I when

a. only the compressive part of the concrete is used;

b. due allowance is made for the ability of concrete to take some tension;

c. the full section is used;

the following examples are worked out.

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* See his letter to the Editor in the Eng. News of Jan. 13, 1910..

EXAMPLE 1 Consider section of beam as shown in Fig. 19. Assume $n = 15$. The three cases are worked out in tabular form on page 55. For cases 1b, 2b, and 3b in this and in the following examples, depths of concrete sections effective for I were obtained by assuming a tensile stress in the concrete of about 200 lbs. per sq. in., and in the reinforcing steel of 16000 lbs. per sq. inch.

EXAMPLE 2 Consider section of beam as shown in Fig. 20. Assume $n = 15$. The three cases are worked out in tabular form on page 56.

EXAMPLE 3. Consider section of beam as shown in Fig. 21. Assume $n = 15$. The three cases are worked out in tabular form on page 57.

EFFECT OF VARYING THE AMOUNT OF TENSILE REINFORCEMENT.

With smaller amounts of tensile steel in the section, that is with smaller p, the neutral axis will be nearer the compressive face and therefore a smaller part of the concrete section will be effective for I. Hence the ratio of I obtained by using the full section to I obtained by using the uncracked section, will increase. With higher percentages of steel the reverse will be the case.

EFFECT OF VARYING p'

Increasing p' brings the neutral axis closer to the compressive face. Accordingly a smaller part of the section remains effective for I and the ratio of I of the full section to I of the uncracked section **increases**. **Decreasing** p' decreases this ratio.

EFFECT OF T FLANGES.

Adding T flanges to the beam brings the neutral axis nearer the compressive face of the section. It therefore decreases the section effective for I and increases the ratio of I of the full section to I of the uncracked section.

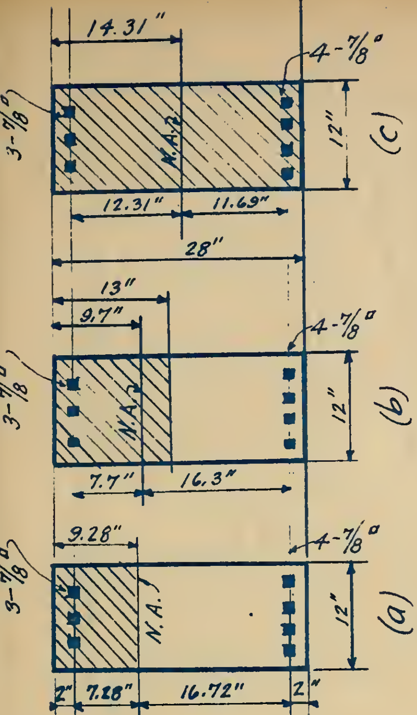

Fig. 20

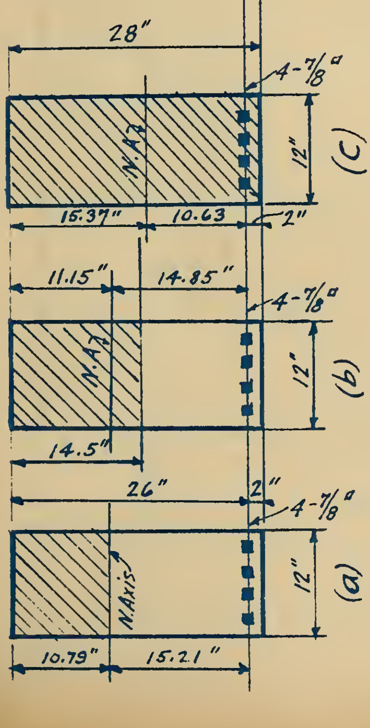

Fig. 19

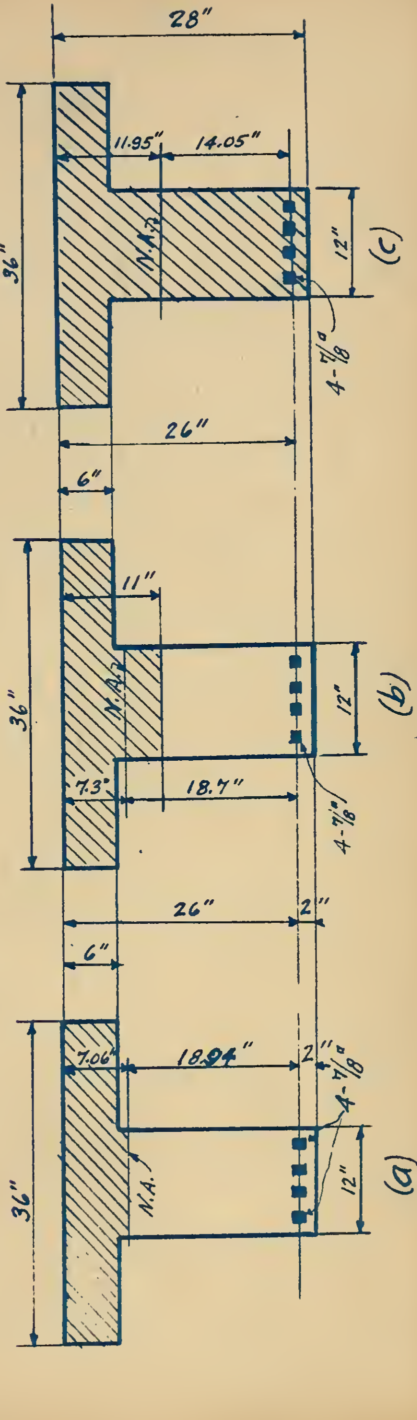

Fig. 21

EXAMPLE I.

Case	1a	1b	1c
Figure	19a	19b	19c
p	$\frac{4 \times .766}{12 \times 26} = .0098$	.0098	.0098
Effective Area of Concrete		$12 \times 14\frac{1}{2} =$ 174 □"	$12 \times 28 = 336$ □"
" " " Steel		$3.06 \times 15 =$ 46 □"	$3.06 \times 14 = 42.8$ □"
" " Total		220 □"	378.8 □"
Mom. of Concrete abt. top		$12 \times 14\frac{1}{2} \times 7\frac{1}{4} = 1262$	$12 \times 28 \times 14 = 4704$
" " Steel " "		$15 \times 3.06 \times 26 = 1192$	$14 \times 3.06 \times 26 = 1114$
Total Mom.	$K = .415$	2454	5818
Distance of c. of gr. from top	$Kd = 10.79''$	$2454 \div 220 = 11.15''$	$5818 \div 378.8 = 15.37''$
I of Concrete	$\frac{1}{3} \times 12 \times 10.79^3 = 5015$	$\frac{1}{12} \times 12 \times 14.5^3 = 3050$	$\frac{1}{12} \times 12 \times 28^3 = 21960$
" " "		$12 \times 14.5 \times 3.90^2 = 2645$	$12 \times 28 \times 1.37^2 = 630$
" " Steel	$15 \times 3.06 \times 15.21^2 = 10625$	$15 \times 3.06 \times 14.85^2 = 10120$	$14 \times 3.06 \times 10.63^2 = 4850$
Total I	15640"^{4}	15815	27440
Comparison	98.8%	100%	173.8%

EXAMPLE 2

Case	2a	2b	2c
Figure	20a	20 b	20 c
p	$4\times.766/12\times26 = .0098$	.0098	.0098
p' ; d'/d	$3/4\times.0098=.00737$; $2/26=.077$	.00737 ; .077	.00737 ; .077
Effective Area of Concrete		$12\times13 =$ 156 □″	$12\times28 =$ 336 □″
" " " Steel		$3.06\times15 =$ 46	$14\times3.06 =$ 42.8
" " " Compr. Steel		$2.3\times14 =$ 32	$14\times2.3 =$ 32.2
" " Total		234 □″	411 □″
Moment of Concrete abt. top		$12\times13\times6\tfrac{1}{2} =$ 1014	$12\times28\times14 =$ 4704
" " Steel " "		$15\times3.06\times26 =$ 1192	$14\times306\times26 =$ 1114
" " Compr. Steel " "		$14\times2.3\times2 =$ 64	$14\times2.3\times2 =$ 64
Total Moment	($K = .357$)	2270	5882
Distance of c. of gr. from top	$kd = 9.28''$	9.7″	14.31″
I of Concrete	$1/3\times12\times9.28^3 =$ 3205	$1/12\times12\times13^3 =$ 2197	$1/12\times12\times28^3 =$ 21960
" " "		$12\times13\times3.2^2 =$ 1598	$12\times28\times.31^2 =$ 32
" " Steel	$3.06\times15\times16.72^2 =$ 12820	$3.06\times15\times16.3^2 =$ 12200	$14\times3.06\times11.69^2 =$ 5850
" " Compr. Steel	$2.30\times14\times7.28^2 =$ 1710	$2.3\times14\times7.7^2 =$ 1910	$14\times2.3\times12.31^2 =$ 4880
Total I	$17735 in^4$	$17905''^4$	$32722''^4$
Comparison	99.1 %	100%	182.8%

EXAMPLE 3

Case	3a	3b	3c
Figure	21 a	21 b	21 c
p	$3.06/36 \times 26 = .00327$	.00327	.00327
Area of Concrete in Flanges		$36 \times 6 = 216\,\square''$	$36 \times 6 = 216\,\square''$
Effective Area of Concrete in Web		$12 \times 5 = 60$	$12 \times 22 = 264$
" " " Steel		$3.06 \times 15 = 45.9$	$14 \times 3.06 = 42.8$
" " Total	$(t/d = 6/26 = .231)$	$321.9\,\square''$	$522.8\,\square''$
Mom. of Flange Area abt. top		$36 \times 6 \times 3 = 648$	$36 \times 6 \times 3 = 648$
" " Web " " "		$12 \times 5 \times 8\frac{1}{2} = 510$	$12 \times 22 \times 17 = 4485$
" " Steel " "		$15 \times 3.06 \times 26 = 1194$	$14 \times 3.06 \times 26 = 1113$
Total Moment	$(K = .272)$	2352	6246
Distance of c. of gr. from top	$Kd = 7.06''$	$7.3''$	$11.95''$
I of Flange Concrete	$1/12 \times 36 \times 6^3 = 648$	$1/12 \times 36 \times 6^3 = 648$	$1/12 \times 36 \times 6^3 = 648$
" " " "	$36 \times 6 \times 4.06^2 = 3555$	$36 \times 6 \times 4.3^2 = 3992$	$36 \times 6 \times 8.95^2 = 17310$
" " Web "	$1/3 \times 12 \times 1.06^3 = 5$	$1/12 \times 12 \times 5^3 = 125$	$1/12 \times 12 \times 22^3 = 10660$
" " " "		$12 \times 5 \times 1.2^2 = 86$	$12 \times 22 \times 5.05^2 = 6730$
" " Steel	$3.06 \times 15 \times 18.94^2 = 16490$	$15 \times 3.06 \times 18.7^2 = 16070$	$14 \times 3.06 \times 14.05^2 = 8460$
Total I	$20698''^4$	$20921''^4$	$43808''^4$
Comparison	99.6%	100%	209.5%

These three examples and the discussion show that:

1. to neglect the part of the concrete taking tension dees not lead to any appreciable errors,*
2. to use the full section as effective for I gives results which are about twice as large as those obtained by using the uncracked section.

It was shown before that the moments at the joints of a frame depend on the ratio of I of the beams to I of the columns, and that the full section may be used to obtain I of a column. Hence if I of the section of maximum stress be considered the criterion of the moment of inertia of the beam, then the ratio of I of the beam to I of the column when using the full beam section will be about two times as large as that ratio when using the uncracked beam section. Doubling the ratio will produce large changes in the distribution of moments in the frame.

The foregoing conclusions as to the part of the concrete section effective for I applies to the section of maximum stress. Under smaller stresses, the cracked area will not be so large, and the ratio of I of the full section to I of the uncracked section will be less than two.

A restrained beam under load has three points of maximum stress: at the center and at the two supports. Between these points the stress gradually diminishes until it becomes zero at the points of contraflexure.

Thus at the center and at the supports the section effective for I is a minimum. As the section moves toward the point of contraflexure a larger part of it becomes effective for I ; near the point of contraflexure the full section is effective. In order to find the I of a beam, it is necessary to evaluate the relative influence of the full sections near the point of contraflexure and of the cracked sections at and near the points of maximum stress.

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* See, also R. E. Spaulding's Letter to the Editor in the Eng. News, Jan. 13, 1910.

EFFECT OF A HAUNCH NEAR THE SUPPORT.

It is the usual practice in designing restrained beams to provide for haunches at the supports. The reasons for the haunches are:

a. the bending moment at the support is sometimes larger than at the center,

b. the center section may have T flanges,

c. the section at the support may be strong enough for bending moment, but not for shear.

To find the effect of the increased section at the support, example 4 is worked out.

EXAMPLE 4 Let the center section be the same as in example 1, page 53. Assume the section at the support to be 5" deeper., and to have to resist a moment 50% higher than the moment at the center. In example 1, $K = .415$, hence $j = .862$ Assume $f_c = 750$ lbs per sq. in. Then the resisting moment of the section

$$= \tfrac{1}{2} \times 750 \times .415 \times .862 \times 12 \times 26 \times 26 = 1089000 \text{ in. lbs.}$$

Assume the resisting moment at the support = 1089000 x 1.5 = about 1500000 in. lbs. The section is as shown in Fig. 22 and the work is tabulated on page 60.

In comparing the results of example 4b with those of example 1b it is noticed that the reinforcing steel changed from 3.06 sq. in. in example 1b to 3.53 sq. in. in example 4b (15.5% increase), the depth of the section changed from 28 in. to 33 in. (17.9% increase), but I changed from 15815 in. to 26180 in. (65% increase). This shows that a 15-20% increase in the section, increases I considerably.

EFFECT OF SECTION NEAR POINT OF CONTRAFLEXURE.

The sections near the point of contraflexture have stresses which do not reach the ultimate tensile strength of concrete. Hence the full section is effective for I. All other things being equal, the I of such sections is about twice the value of I of a section under maximum stresses. (Compare results in previous examples).

EXAMPLE 4

Case	4a	4b	4c
Figure	22	22	22
p (R = 133.8)	$3.53/12\times31 = .0095$	.0095	.0095
Effective Area of Concrete		$12\times17 =$ 204□"	$12\times33 =$ 396□"
" " " Steel		$15\times3.53 =$ 52.9	$14\times3.53 =$ 49.4
" " Total		256.9□"	445.4□"
Mom. of Concrete abt. Top		$12\times17\times8\frac{1}{2} =$ 1734	$12\times33\times16.5 =$ 6540
" " Steel " "		$15\times3.53\times31 =$ 1641	$14\times3.53\times31 =$ 1532
Total Moment	(K = .414)	3375	8072
Distance of c.of gr. from top	12.85"	13.15"	18.1"
I of Concrete	$1/3\times12\times12.85^3 =$ 8480	$1/12\times12\times17^3 =$ 4910	$1/12\times12\times33^3 =$ 35950
" " "		$12\times17\times4.65^2 =$ 4410	$12\times33\times1.6^2 =$ 1010
" " Steel	$15\times3.53\times18.15^2 =$ 17470	$15\times3.53\times17.85^2 =$ 16860	$14\times3.53\times12.9^2 =$ 8230
Total I	$25950''^4$	$26180''^4$	$45190''^4$
Comparison	99.2 %	100 %	173 %

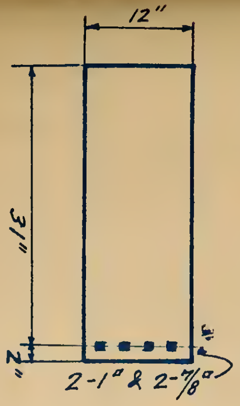

Fig. 22

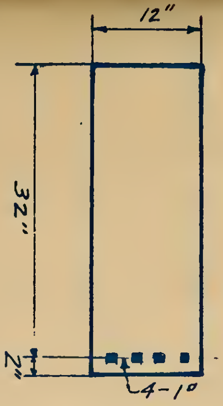

Fig. 23

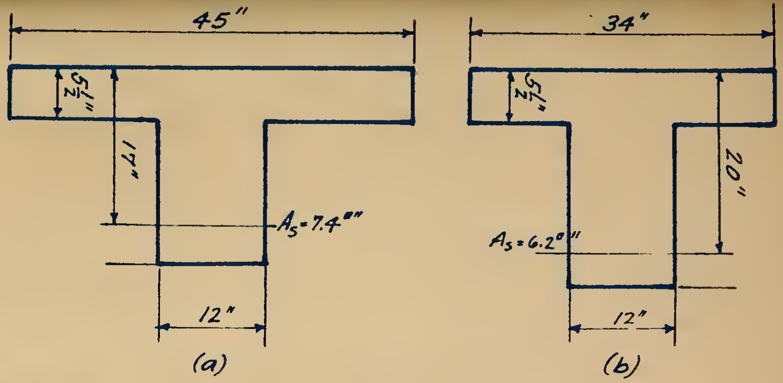

Fig. 24

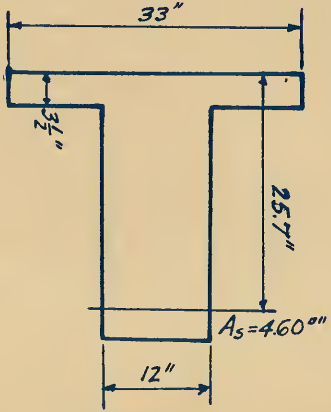

Fig. 24 c

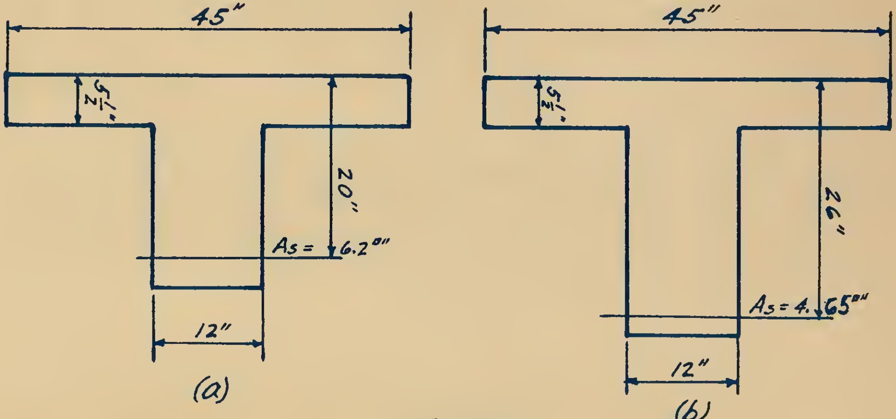

Fig. 25

EFFECT OF T. ACTION

Beams are often designed with T flanges to help take compression. The T flanges influence the I of the section by

1. concentrating the compressive area nearer the end of the beam, thus increasing I,
2. decreasing the depth of the beam, which decreases I and
3. increasing the amount of reinforcing steel, which increases I..

The following examples are worked out in order to find the combined influence on I of all these factors. The I's of T beams are compared with the I of a simple beam of the same resisting moment.

Assume a simple beam as shown in Fig. 23, $p = \frac{4 \times 1}{12 \times 32} = .0104$, $n = 15$, $k = .424$, $j = .859$, $f_c = 750$ lbs. per sq. in., $kd = 13.6$ in.

Resisting moment $= \frac{1}{2} \times 750 \times .424 \times .859 \times 12 \times 32 \times 32 = 1676000$ in. lbs. $f_s = \frac{1676000}{4 \times .859 \times 32} = 15240$ lbs. per sq. in.

The moment of inertia of the section is

$$1/3 \times 12 \times 13.6 \times 13.6 \times 13.6 = 10000$$
$$4 \times 15 \times 18.4 \times 18.4 = \underline{20300}$$
$$I = 30300 \text{ in.}^4$$

Example 5 The T beam has the same stresses as the simple beam.

Case a. $t = 5\frac{1}{2}$ in., $b = 45$ in.
Case b. $t = 5\frac{1}{2}$ in., $b = 34$ in.
Case c. $t = 3\frac{1}{2}$ in., $b = 33$ in.

EXAMPLE 6 Concrete stresses in T beam smaller than in simple beam.

Case a. $t = 5\frac{1}{2}$ in., $b = 45$ in., $d = 20$ in.
Case b. $t = 5\frac{1}{2}$ in., $b = 45$ in., $d = 26$ in.

The cross sections for examples 5 and 6 are shown in Figs. 24 and 25. The work is tabulated on page 63.

	EXAMPLE 5			EXAMPLE 6	
Case	5a	5b	5c	6a	6b
Figure	24a	24b	24c	25a	25b
t/d	.324	.275	.136	.275	.211
p	.00968	.00913	.0054	.0069	.00402
k (from tables)	.424	.424	.424	.372	.304
j (" ")	.875	.89	.93	.89	.91
M	1676000 "#	1676000 "#	1676000 "#	1676000 "#	1676000 "#
$f_s\ (=\frac{M}{A_s jd})$	15240 #	15200 #	15240 #	15200 #	15240 #
$1-t/2kd$	.618	.676	.839	.631	.652
$f_c\ [=\frac{M}{btjd(1-t/2kd)}]$	740 #	745 #	730 #	600 #	440 #
kd	7.2"	8.5"	10.9"	7.44"	7.9"
I of Flange	$\frac{1}{12}\times45\times5.5^3=$ 625	$\frac{1}{12}\times34\times5.5^3=$ 470	$\frac{1}{12}\times33\times(3\frac{1}{2})^3=$ 120	$\frac{1}{12}\times45\times5.5^3=$ 620	$\frac{1}{12}\times45\times5.5^3=$ 620
" " "	$45\times5.5\times4.45^2=$ 4900	$34\times5\frac{1}{2}\times5.75^2=$ 6180	$33\times3.5\times9.15^2=$ 9670	$45\times5.5\times4.69^2=$ 5450	$45\times5.5\times5.15^2=$ 6560
" " Web	$\frac{1}{3}\times12\times1.7^3=$ 20	$\frac{1}{3}\times12\times3^3=$ 110	$\frac{1}{3}\times12\times7.4^3=$ 1620	$\frac{1}{3}\times12\times1.94^3=$ 30	$\frac{1}{3}\times12\times2.4^3=$ 60
" " Steel	$7.4\times15\times9.8^3=$ 10675	$6.2\times15\times11.5^2=$ 12310	$4.6\times15\times14.8^2=$ 15100	$6.2\times15\times12.56^2=$ 14670	$4.65\times15\times18.1^2=$ 22900
Total I	16220 "4	19070 "4	26510 "4	20770 "4	30140 "4
Comparison with Simple Beam, Fig. 23, page 62.	53.6 %	62.9 %	87.5 %	68.6 %	99.6 %

These examples show that when a beam utilizes the full value of the T flanges, that is when their compressive stress is equal to that of a simple beam with the same resisting moment, then I is reduced considerably. The longer the T flanges (when fully utilized), the smaller d and hence I. With a thinner slab, d is not reduced so much and I is raised somewhat. When, however, the full value of the T flanges is not used, then I may be increased until it equals (or even exceeds) the I of a simple beam with the same resisting moment.

Other variables entering in the design of T beams will not change these conclusions.

C. APPLICATION TO FIXED BEAMS.

The moments at the supports of a fixed beam depend on I of the beam. Ordinary formulas usually assume a uniform I. But, as was shown before, I varies throughout the length of the beam. By means of the equations derived in Chapter V, it is possible to study the effect of varying I on the moments at the supports of a fixed beam.

EXAMPLE 7

Let A B be a fixed beam with a uniform load of w lbs. per ft. Assume a variation of I as shown in Fig. 26 which is in accordance with the conclusions drawn from previous examples.

The moment at the support of a fixed beam is (from chapter V)

$$C_{AB} = \frac{-\frac{F}{I}\int\frac{x^2dx}{I} + \frac{F\bar{x}}{I}\int\frac{xdx}{I}}{\int\frac{dx}{I}\int\frac{x^2dx}{I} - \int\frac{xdx}{I}\int\frac{xdx}{I}}$$

where $\frac{F}{I}$ is the area of the $\frac{M}{I}$ diagram of a simple beam with the same load and $\frac{F\bar{x}}{I}$ is the moment of that area about A.

The moment at any point due to the load w lbs. per ft. $= \frac{wlx}{2} - \frac{wx^2}{2}$.

$\frac{M}{I}$ at any point $= \frac{wlx}{2I} - \frac{wx^2}{2I}$. The area of the $\frac{M}{I}$ diagram $= \int \frac{wlx\,dx}{2I} - \int \frac{wx^2dx}{2I}$.

The moment of this area about A $= \int \frac{wlx^2dx}{2I} - \int \frac{wx^3dx}{2I}$

$$\text{Hence } C_{AB} = \frac{\cancel{-\int \frac{wlx\,dx}{2I}\int \frac{x^2dx}{I}} + \int \frac{wx^2dx}{2I}\int \frac{x^2dx}{I} + \cancel{\int \frac{wlx^2dx}{2I}\int \frac{x}{I}dx} - \int \frac{wx^3dx}{2I}\int \frac{x\,dx}{I}}{\int \frac{dx}{I}\int \frac{x^2dx}{I} - \int \frac{x\,dx}{I}\int \frac{x\,dx}{I}} = \frac{\int \frac{x^2dx}{I}\int \frac{x^2dx}{I} - \int \frac{x^3dx}{I}\int \frac{x\,dx}{I}}{\int \frac{dx}{I}\int \frac{x^2dx}{I} - \int \frac{x\,dx}{I}\int \frac{x\,dx}{I}} \cdot \frac{w}{2}$$

The work is tabulated on page 66. When uniform I is assumed, $C_{AB} = -\frac{wl^2}{12} = -12w$. Hence, considering varying I gives a moment at the support $\frac{13.5-12}{12}$ or 12.5% higher than the usual value. Since the haunch has a higher value of I than the middle part, the moment at the support will always be higher than $\frac{wl^2}{12}$.

EXAMPLE 8

Let A B be a fixed beam with a concentrated load P acting at the middle of the beam. The variation of I is shown in Fig. 27. The moment at any point to the left of the load $= \frac{Px}{2}$; to the right of the load $= \frac{Px}{2} - P(x - l/2)$. $\frac{M}{I}$ at any point to the left of the load $= \frac{Px}{2I}$; to the right of the load $= \frac{Px}{2I} - \frac{P}{I}(x - l/2)$. The area of the $\frac{M}{I}$ diagram $= \int_0^l \frac{Px\,dx}{2I} - \int_{l/2}^{l} \frac{P}{I}(x - l/2)dx$. The moment of this area about A $= \int_0^l \frac{Px^2dx}{2I} - \int_{l/2}^{l} \frac{Px}{I}(x - l/2)dx$. Hence

$$C_{AB} = \frac{\int \frac{x^2dx}{I}\int_{l/2}^{l} \frac{P}{I}(x - \frac{l}{2})dx - \cancel{\int \frac{x^2dx}{I}\int \frac{Px\,dx}{2I} + \int \frac{x\,dx}{I}\int \frac{Px^2dx}{2I}} - \int \frac{x\,dx}{I}\int_{l/2}^{l} \frac{Px}{I}(x - l/2)dx}{\int \frac{dx}{I}\int \frac{x^2dx}{I} - \int \frac{x\,dx}{I}\int \frac{x\,dx}{I}} = \frac{\int \frac{x^2dx}{I}\int_{l/2}^{l} \frac{(x - l/2)dx}{I} - \int \frac{x\,dx}{I}\int_{l/2}^{l} \frac{x(x - l/2)dx}{I}}{\int \frac{dx}{I}\int \frac{x^2dx}{I} - \int \frac{x\,dx}{I}\int \frac{x\,dx}{I}} P$$

The work is tabulated on page 66. Comparing the value found on page 66 with the usual value of $\frac{Pl}{8}$ or $1.5P$ $(l=12)$, an increase of $\frac{1.75-1.5}{1.5} = 16.7\%$ is found.

	EXAMPLE 7				EXAMPLE 8					EXAMPLE 9				
X	$\frac{1}{I}$	$\frac{X}{I}$	$\frac{X^2}{I}$	$\frac{X^3}{I}$	$\frac{1}{I}$	$\frac{X}{I}$	$\frac{X^2}{I}$	$\frac{X-\ell/2}{I}$	$\frac{X(X-\frac{\ell}{2})}{I}$	$\frac{1}{I}$	$\frac{X}{I}$	$\frac{X^2}{I}$	$\frac{X-a}{I}$	$\frac{X(X-a)}{I}$
1	.5	.5	.5	.5	.5	.5	.5	—	—	.5	.5	.5	–	–
3	.5	1.5	4.5	13.5	.5	1.5	4.5	—	—	1	3	9	–	–
5	1	5	25	125	1	5	25	—	—	1	5	25	1	5
7	1	7	49	343	1	7	49	1	7	.5	3.5	24.5	1.5	10.5
9	.5	4.5	40.5	364.5	.5	4.5	40.5	1.5	13.5	.5	4.5	40.5	2.5	22.5
11	.5	5.5	60.5	665.5	.5	5.5	60.5	2.5	27.5	1	11.	121	7	77
Σ	4	24	180	1512	4	24	180	5	48	4.5	27.5	220.5	12	115
$\Delta X \cdot \Sigma$	8	48	360	3024	8	48	360	10	96	9	55	441	24	230
C_{AB}	$\frac{\overset{36}{\cancel{360}} \times \overset{30}{\cancel{360}} - 3024 \times \cancel{48}}{\cancel{8} \times \underset{60}{\cancel{360}} - 48 \times \cancel{48}} \frac{w}{2} = -13.5\,w$ $\left(= \frac{w\ell^2}{10.67}\right)$				$\frac{360 \times 10 - 96 \times 48}{8 \times 360 - 48 \times 48} P = -1.75\,P$ $\left(= \frac{P\ell}{6.85}\right)$					$\frac{441 \times 24 - 230 \times 55}{9 \times 441 - 55 \times 55} P = -2.20\,P$				

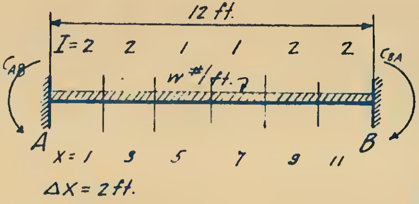

Fig. 26

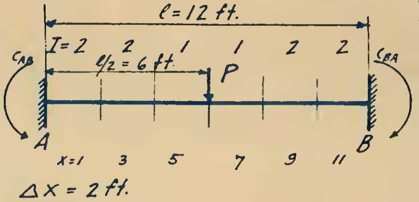

Fig. 27

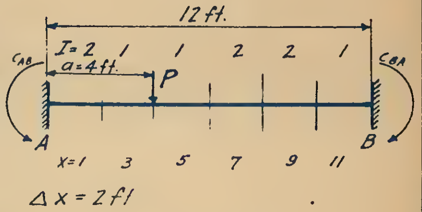

Fig. 28

EXAMPLE 9

Let A B be a fixed beam with a concentrated load P placed 4 ft. from the support A. Let the variation of I be as in Fig. 28. The moment at any point to the left of the load $= \frac{(\ell-4)}{\ell}Px$; to the right of the load $= \frac{(\ell-4)}{\ell}Px - P(x-4)$. The $\frac{M}{I}$ at any point to the left of the load $= \frac{(\ell-4)}{\ell}\frac{Px}{I}$; to the right of the load $= \frac{(\ell-4)}{\ell}\frac{Px}{I} - \frac{P(x-4)}{I}$. The area of the $\frac{M}{I}$ diagram $= \int_0^\ell \frac{(\ell-4)}{\ell}\frac{Px\,dx}{I} - \int_4^\ell \frac{P(x-4)}{I}dx$. The moment of this area about A $= \int_0^\ell \frac{(\ell-4)}{\ell}\frac{Px^2dx}{I} - \int_4^\ell \frac{Px(x-4)}{I}dx$. Hence

$$C_{AB} = \frac{\int \frac{x^2dx}{I}\int_4^\ell \frac{P(x-4)dx}{I} - \int\frac{x^2dx}{I}\int\cancel{\frac{(\ell-4)}{\ell}\frac{Px\,dx}{I}} + \int\frac{x\,dx}{I}\int\cancel{\frac{(\ell-4)}{\ell}\frac{Px^2dx}{I}} - \int\frac{x\,dx}{I}\int_4^\ell\frac{Px(x-4)dx}{I}}{\int\frac{dx}{I}\int\frac{x^2dx}{I} - \int\frac{x\,dx}{I}\int\frac{x\,dx}{I}} = \frac{\int\frac{x^2dx}{I}\int_4^\ell\frac{(x-4)dx}{I} - \int\frac{x\,dx}{I}\int_4^\ell\frac{x(x-4)dx}{I}}{\int\frac{dx}{I}\int\frac{x^2dx}{I} - \int\frac{x\,dx}{I}\int\frac{x\,dx}{I}}\cdot P$$

The work is tabulated on page 66 .

The value of the moment at the support assuming a uniform I $= -\frac{Pa(\ell-a)^2}{\ell^2}$ or $-1.78P\,(a=4)$. Hence the moment found on page 66 is $\frac{2.2-1.78}{1.78} =$ 23.6% higher than the value found in the usual way.

These examples, where the variations of I were taken at about what they would be for the average case, show that the moment coefficients used for fixed beams are too low when applied to concrete beams. This includes all fixed concrete beams having haunches at the supports.

Studies of the effect of haunches on the distribution of moments were made in several cases. They all assume one value of I to be constant from the supports to the points of contraflexure and another between the points of contraflexure.

P. E. Stevens * works out curves giving the ratio of the moment at the center to the moment at the support in terms of the ratio of I at the center to I at the support. He assumes the case of the middle span of a beam continuous over many supports loaded with a uniform load. It is interesting to note that he assumes the I of a section to be proportional to the amount of steel in the section. He thus gives no consideration to the influence on I of an increase in depth.

Prof. R. E. Spaulding ** works out two cases:

1. Beam fixed at both ends.
2. Beam fixed at one end and simply supported at the other.

In both cases one value of I is assumed for sections between supports and points of contraflexure and another value for sections between the points of contraflexure. Prof. Spaulding plots curves showing the value of the moment at the support and the moment at the center for varying ratios of I at the support to I at the center. These curves are reprinted in G. A. Hool's "Reinforced concrete construction Vol. 2, page 391 and in G. A. Hool and N. C. Johnson's "Concrete Engineers' Handbook", page 340. Prof. R. E. Spaulding states that his results can be applied to conditions met with in building construction.

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* In the Trans. A. S. C. E., vol. 60, page 496

** ""Moments in Continuous Reinforced Concrete Beams under Uniform Loading", by Prof. R. E. Spaulding, Eng. News, Sept. 30, 1909. Also Prof. R. E. Spaulding's Letter to the Editor in the Eng. News, Jan. 13, 1910.

These curves (Mr. Stevens' and Prof. Spaulding's) do not cover the subject completely because

1. they do not consider concentrated loads,
2. they do not take care of unsymmetrical beams since they assume the I of both haunches *equal*.

D. APPLICATION TO FRAMES.

Just as the moment at the support of a fixed beam is influenced by unequal I's at the haunch and at the center, so is the moment due to a rigid joint. The equations derived in Chapter V can be applied to any case of a beam or a column when the variation of I is known.

It is desirable, however, to establish short-cuts and test their degree of correctness by the equations of Chapter V.

Usually there are points of maximum and minimum moments in beams and columns. At the point of maximum moment, the section effective for I is the smallest; at minimum moment, the full section is probably effective. Thus a section at the haunch of a beam has only a part of it effective for I. A section in the middle of a beam is also only partly effective; it usually has less depth than the section at the haunch, or it has T flanges; hence its I is less than that of the haunch section. At the point of contraflexture the full section is effective and its I is greater than that of the center and possibly that of the haunch.

But their influence on the I of the beam is not the same. In going back to Chapter V , it is noticed that the equations are based on $\frac{d^2y}{dx^2} = -\frac{M}{EI}$

Thus the moments at the joints depend on $\frac{M}{I}$ . . The moment at the point of contraflexure, where the full section is used for I, is zero; near that point it is very small. Hence $\frac{M}{I}$ at and near the point of contraflexure is very small and will influence the results only to a small extent. $\frac{M}{I}$ at the haunches or at the center, where the moment is a maximum, has a large influence on the resultant I of the beam. This gives a basis for reducing tho number of variations of I it is necessary to consider. Mr. S. E. Thompson states * in a letter to the Editor that considering the variations of I from point to point, neglecting the cracked parts of the sections, hence considering I a maximum at the points of contraflexure and from there as varying to a minimum at points of largest moments, he obtained results quite close to those of Prof. R.E. Spaulding who used only one value of I from supports to points of contraflexure and another between points of contraflexure.

Hence the only place in a beam where the full section can be used for I has little influence on the resultant I of the beam.

An examination of the equations of Chapter V will show that I enters into the denominator of the various summations. As they stand, none of these summations can be canceled out of the expression; nor can that be done with $\frac{1}{I}$. That is, a value of $\frac{1}{I}$ assumed so as to make $\frac{1}{I}\int x\,dx = \int \frac{x\,dx}{I}$, will not necessarily make $\frac{1}{I}\int x^2dx = \int \frac{x^2dx}{I}$, nor $\frac{1}{I}\int dx = \int \frac{dx}{I}$.

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* Eng. News, Jan. 13, 1910.

Again a value of $\frac{1}{I}$ such as to make $\frac{1}{I}\int x^2dx = \int \frac{x^2dx}{I}$, will *not* make $\frac{1}{I}\int dx = \int \frac{dx}{I}$, nor $\frac{1}{I}\int x dx = \int \frac{x dx}{I}$. Hence no value of $\frac{1}{I}$ (and therefore of I) can be found which can be taken out of the summation. This means that no exact average value of I for a beam with a varying I is possible.

The following examples will show how close the moments obtained by several methods of approximating the average I come to the actual moments as obtained by the modified slope deflection equations (Chapter V)

EXAMPLE 10

To find the moment at the joint A of an unloaded beam A B The variation of I is shown in Fig. 29.

The point of contraflexure is in the middle part of the beam. In that part the full section is effective for I, hence its I was taken at twice the value of I of the other part.

Case a. Use the modified slope deflection equations (Chapter V). This case gives the moment at the joint when due consideration is given to the variation of I through the beam.

Case b. Assume I at the point of contraflexure to be the same as at the support, thus giving a uniform I. This case shows how a change of I at the point of contraflexure influences the moment at the joint.

Case c. Use I of the full section. This will make I at the support about twice its original value or equal to I at the point of contraflexure.

The work is shown on page 72.

A comparison of the results of the three cases shows that assuming at the point of contraflexure only a part of the section effective for I gives results very close to the actual.

EXAMPLE 10

Case a. Fig. 29.

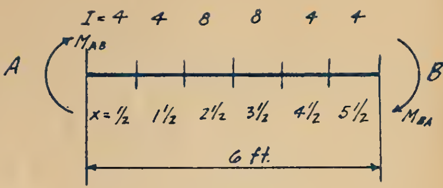


Fig. 29

Using modified slope deflection equations. $\Delta X = 1$ ft.

$$\Sigma \frac{1}{I} = (1/4 + 1/4 + 1/8 + 1/8 + 1/4 + 1/4) = 1.25$$

$$\Sigma \frac{x}{I} = (1/8 + 3/8 + 5/16 + 7/16 + 9/8 + 11/8) = 3.75$$

$$\Sigma \frac{x^2}{I} = (1/16 + 9/16 + 25/32 + 49/32 + 81/16 + 121/16) = 249/16 = 15.56$$

Denominator of M_{AB} (See page 44) $= \int\frac{dx}{I}\int\frac{x^2dx}{I} - \int\frac{xdx}{I}\int\frac{xdx}{I}$

$$= 1.25 \times 15.56 - 3.75^2 = 19.47 - 14.07 = 5.4$$

Numerator of $M_{AB} = E\left[\theta_A\int\frac{x^2dx}{I} + \theta_B\left(\ell\int\frac{xdx}{I} - \int\frac{x^2dx}{I}\right) - d\int\frac{xdx}{I}\right]$ (page 44)

$$= E[15.56\,\theta_A + (22.5 - 15.56)\theta_B - 3.75\,d]$$

$$\therefore\ M_{AB} = E[2.88\,\theta_A + 1.29\,\theta_B - .69\,d]$$

Case b

I is uniform and equals 4.

$$M_{AB} = \frac{2EI}{\ell}\left(2\theta_A + \theta_B - \frac{3d}{\ell}\right) = \frac{8}{6}E(2\theta_A + \theta_B - .5d) = E(2.67\,\theta_A + 1.33\,\theta_B - .67d)$$

Case c.

I is uniform and equals 8.

$$M_{AB} = \frac{16}{6}E(2\theta_A + \theta_B - .5d) = E(5.33\,\theta_A + 2.67\,\theta_B - 1.33\,d)$$

Using the full section throughout changes the moment at the joint considerably. Note that it is impossible to find a value of I for 10a to make the expression inside the parenthesis $= 2\theta_A + \theta_B - \frac{3d}{\ell}$. This is another proof of the impossibility of finding an exact average value of I for a beam with a varying I.

EXAMPLE 11.

To show influence of I at points of maximum moment and minimum moment on I of the beam. Let A B be an unloaded beam restrained at the ends. The variation of I is as shown in Fig. 30 a. The I's are assumed such as to give the usual ratios of I at support (part section) to I at point of contraflexure. (full section) with variations between these extremes. Note that the beam is not symmetrical about its middle. For an unloaded beam the point of contraflexure is in the middle part of the beam.

Case a. Modified equations, actual variation of I.

Case b. I near point of contraflexure reduced 50%.

This will show the influence on the moment at the joint of assuming only part of the section effective for I at point of contraflexure.

Case c. I near supports increased 75% to show influence on I of beam due to sections near supports. This increase makes I almost equal to I of a full section.

Case d. To show influence of I at supports, I at each support is assumed constant over ½ of the beam.

Case e. The average of the various $\frac{1}{I}$'s of the beam, was inverted and used as a uniform I.

Case f. The average of the $\frac{1}{I}$ values at the supports was inverted and used as a uniform I.

EXAMPLE 11

Case	11a			11b			11c			11d			11e	11f
Figure	30a			30b			30c			30d			—	—
	$\frac{1}{I}$	$\frac{x}{I}$	$\frac{x^2}{I}$	$\frac{1}{I}$	$\frac{x}{I}$	$\frac{x^2}{I}$	$\frac{1}{I}$	$\frac{x}{I}$	$\frac{x^2}{I}$	$\frac{1}{I}$	$\frac{x}{I}$	$\frac{x^2}{I}$	$\frac{1}{I}$	$\frac{1}{I}$
$x = 1/2$	.25	.125	1/16	.25	.125	1/16	.143	.071	1/28	.25	.125	1/16	1/4 = .25	1/4
" $1\frac{1}{2}$	.125	.188	9/32	.125	.188	9/32	.125	.188	9/32	.25	.375	9/16	1/8 = .125	1/4
" $2\frac{1}{2}$	.0417	.25	25/96	.083	.5	25/48	.083	.25	25/96	.25	.625	25/16	1/24 = .083	1/4
" $3\frac{1}{2}$	.0416		49/96	.084		49/48			49/96	.2	.7	49/20	1/24 =	1/5
" $4\frac{1}{2}$	.1	.45	81/40	.1	.45	81/40	.1	.45	81/40	.2	.9	81/20	1/10 = .1	1/5
" $5\frac{1}{2}$	.2	1.1	121/20	.2	1.1	121/20	.133	.734	121/30	.2	1.1	121/20	1/5 .2	1/5
$\Delta x.\Sigma$	.758	2.113	9.189	.842	2.363	9.961	.584	1.693	7.146	1.35	3.825	14.738	.76	1.35
Denominator of $M_{AB} = \int\frac{dx}{I}\int\frac{x^2dx}{I} - \int\frac{xdx}{I}\int\frac{xdx}{I}$	$.758 \times 4.189 - 2.113^2$ $= 6.98 - 4.46 = 2.52$			$.842 \times 9.961 - 2.363^2$ $= 8.40 - 5.58 = 2.82$			$.584 \times 7.146 - 1.693^2$ $= 4.18 - 2.87 = 1.31$			$1.35 \times 14.738 - 3.825^2$ $= 19.90 - 14.62$ $= 5.28$			$I = \frac{6}{.76} = 8$ abt.	$I = \frac{6}{1.35} = 4\frac{1}{2}$ abt
Numerator of M_{AB}: $\theta_A\int\frac{x^2dx}{I} + \theta_B(l\int\frac{xdx}{I} - \int\frac{x^2dx}{I}) - d\int\frac{xdx}{I}$ — θ_A =	9.189			9.961			7.146			14.738			$M_{AB} = \frac{2E\cdot 8}{6}(2\theta_A + \theta_B - \frac{3d}{6})$ $= \frac{8}{3}(2\theta_A + \theta_B - \frac{3d}{6})E$ $= E(5.33\theta_A + 2.67\theta_B - 1.33d)$	$M_{AB} = \frac{2E\,4.5}{6}(2\theta_A + \theta_B - \frac{3d}{6})$ $= \frac{9}{6}E(2\theta_A + \theta_B - \frac{3d}{6})$ $= E(3\theta_A + 1.5\theta_B - .75d)$
θ_B =	$6 \times 2.113 - 9.189 = 3.489$			$6 \times 2.363 - 9.961 = 4.217$			$6 \times 1.693 - 7.146 = 3.012$			$6 \times 3.825 - 14738$ $= 8.21$				
d =	-2.113			-2.363			-1.693			-3.825				
M_{AB}/E	$3.65\theta_A + 1.386\theta_B - .838d$			$3.53\theta_A + 1.50\theta_B - .838d$			$5.45\theta_A + 2.29\theta_B - 1.29d$			$2.79\theta_A + 1.56\theta_B - .724d$				
M_{AB}/E	$.168(2.17\theta_A + .83\theta_B - 3\frac{d}{6})$			$.168(2.10\theta_A + .89\theta_B - 3.0\frac{d}{6})$			$.258(2.11\theta_A + .89\theta_B - 3\,d/6)$			$.145(1.93\theta_A + 1.08\theta_B - 3\frac{d}{6})$			2.33( )	1.5( )
Comparison	100%			100%			153.7%			86.4%			139%	89.3%

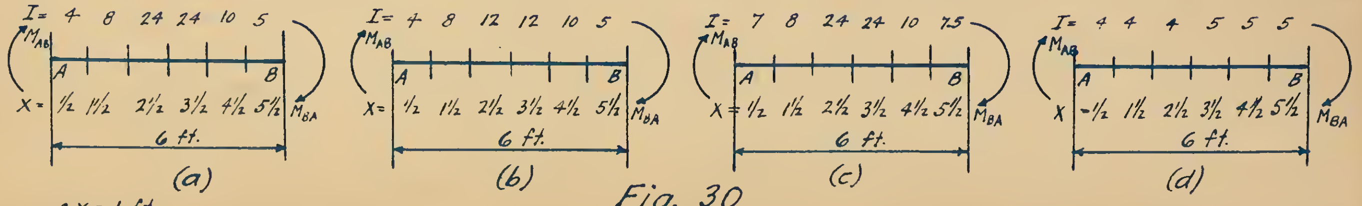


Fig. 30

$\Delta x = 1$ ft.
in all cases

Cases a to d were solved by using the modified slope deflection equations, while cases e and f — by using the ordinary equations. The work is tabulated on page 74. Note again that for cases a to d, the moment equation cannot be put into the form $\frac{2EI}{\ell}(2\theta_A+\theta_B-\frac{3d}{\ell})$; no factor common to the coefficients of θ_A, θ_B and $\frac{d}{\ell}$ can be found.

However, for purposes of comparison a factor was used for each of cases a to d which approximates the above mentioned condition. This factor was found by taking an average of the coefficients of θ_A, θ_B and $\frac{d}{\ell}$.

A comparison of the equations for moments at the joints and of the factors mentioned above shows that using part sections for I . throughout the beam whether by the modified equations or by the ordinary gives results which are about 10% off from the correct results. By using the full section throughout the beam, results are 50-60% off.

EXAMPLE 12.

To show the effect on the moment at the joint of the various shortcuts to the modified slope deflection equations. Let A B C D be a frame (see Fig. 31) symmetrical about a vertical line through the middle of B C, with a load P, acting in the middle of B C. The I's of the columns are uniform, I of B C is varying as shown in Fig. 31.

Case a. Consider I varying as in Fig. 31. Find the moment at the joint B by means of the modified slope deflection equations.

Case b. Assume I of part section throughout. Then for I at the point of contraflexure use the value of I at the haunch. I is uniform and equals 4.

EXAMPLE 12

Let ABCD be a frame symmetrical about a vertical center line. Let BC have a load P acting in its middle. I of AB and CD is uniform and equals 3. To find the moment at joint B.

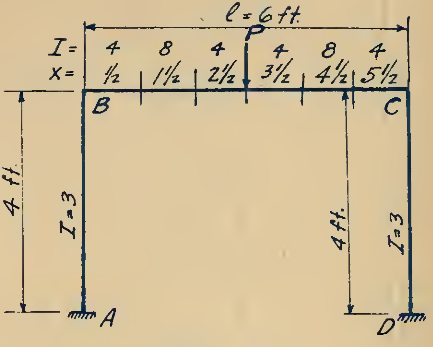


Fig. 31

$\theta_A = 0$, since A is fixed.

No deflection in AB, due to symmetry of frame and load.

$$M_{BA} = \frac{2EI}{l}\left(2\theta_B+\theta_A-\frac{3d}{l}\right) = \frac{6}{4}E(2\theta_B) = 3E\theta_B$$

Case a. I of BC is varying as shown in Fig. 31 $\Delta x = 1$ ft.

$$\Sigma\frac{1}{I} = 2\left(\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{4}\right) = 1.25 \qquad \Sigma\frac{x}{I} = \left(\tfrac{1}{8}+\tfrac{3}{16}+\tfrac{5}{8}+\tfrac{7}{8}+\tfrac{9}{16}+\tfrac{11}{8}\right) = 3.75$$

$$\Sigma\frac{x^2}{I} = \left(\tfrac{1}{16}+\tfrac{9}{32}+\tfrac{25}{16}+\tfrac{49}{16}+\tfrac{81}{32}+\tfrac{121}{16}\right) = 15.06$$

$$\Sigma\frac{x-l/2}{I} = \left(\tfrac{1}{8}+\tfrac{3}{16}+\tfrac{5}{8}\right) = .94 \qquad \Sigma\frac{x(x-l/2)}{I} = \left(\tfrac{7}{16}+\tfrac{27}{32}+\tfrac{55}{16}\right) = 4.72$$

Denominator $= 1.25 \times 15.06 - 3.75^2 = 18.82 - 14.07 = 4.75$

Numerator of Loading Factor (Chapter V) $= (15.06 \times .94 - 3.75 \times 4.72)P$

$= (14.16 - 17.69)P = -3.53P$ $\quad\therefore$ Loading Factor $= -\frac{3.53}{4.75}P = -.743P$

$$\int\frac{x^2dx}{I}\,\theta_B = 15.06\,\theta_B\ ;\quad \left[l\int\frac{xdx}{I}-\int\frac{x^2dx}{I}\right]\theta_C = (6\times 3.75 - 15.06)\,\theta_C = 7.44\,\theta_C$$

Due to symmetry $\theta_C = -\theta_B$ and $d = 0$ for member BC.

Hence $M_{BC} = \frac{E(15.06\,\theta_B - 7.44\,\theta_B)}{4.75} - .743P = E(1.6\,\theta_B) - .743P$

$M_{BA} + M_{BC} = 0$

$\therefore\ 3E\theta_B + 1.6E\theta_B - .743P = 0 \quad ; \quad 4.6E\theta_B = .743P$

$E\theta_B = .1616P \qquad \therefore\ M_{BA} = 3E\theta_B = .485P$

Case b. I of BC is uniform and equals 4

$$M_{BC} = \frac{2EI}{l}\left(2\theta_B+\theta_C-\frac{3d}{l}\right) - \frac{Pl}{8}.$$

But $\theta_C = -\theta_B$ and $d=0$

$\therefore \quad M_{BC} = \frac{8}{6} E(\theta_B) - .75\,P$ $\qquad l=6\ ,\ I=4$

$= 1.33\ E\ \theta_B - .75\ P$

$M_{BA} + M_{BC} = 0 \qquad \therefore \quad 3E\theta_B + 1.33\,E\theta_B - .75P = 0$

$E\theta_B = \frac{.75}{4.33}P = .173\ P$

$M_{BA} = 3E\theta_B = .519\ P$

Case c. $\qquad$ I of BC uniform and equals 8

$M_{BC} = \frac{16}{6}E\theta_B - .75P = 2.67\ \theta_B E - .75\ P$

$M_{BA} + M_{BC} = 0 \qquad 3E\theta_B + 2.67E\theta_B = .75P$

$E\theta_B = \frac{.75P}{5.67} = .1327\ P$

$M_{BA} = 3E\theta_B = .398\ P$

Comparison

	Case a	Case b	Case c
M_{BA} =	.485 P	.519 P	.398 P
Difference		+.034 P	−.087 P
%		+ 7%	− 18%

EXAMPLE 13

Take a frame (Fig. 32) symmetrical about a vertical center line. Let load Q act as shown. I of AB and CD is uniform and equals .8

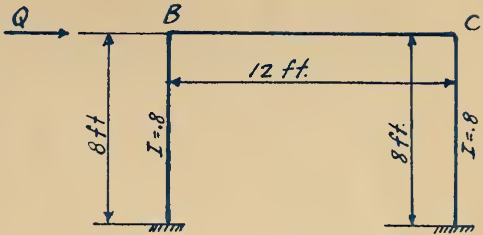


Fig. 32

Due to symmetry $\theta_C = \theta_B$.

$M_{BA} = \frac{2E\times.8}{8}(2\theta_B + \theta_A - \frac{3d}{8})$

$= .2E(2\theta_B - \frac{3d}{8}) = .4E\theta_B - .075Ed$

(Since θ_A and $\theta_D = 0$)

$M_{CD} = \frac{2E\times.8}{8}(2\theta_C - \frac{3d}{8}) = .4E\theta_B - .075Ed$

$M_{AB} = \frac{2E\times.8}{8}(\theta_B - \frac{3d}{8}) = .2E\theta_B - .075Ed$

$M_{DC} = \frac{2E\times.8}{8}(\theta_C - \frac{3d}{8}) = .2E\theta_B - .075Ed$

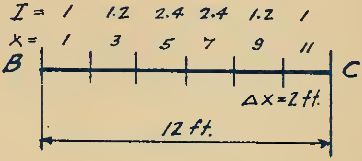


Fig. 32 a.

Consider AB and CD as free members.

$\Sigma M = 0 \quad \therefore \quad 8Q + M_{BA} + M_{AB} + M_{CD} + M_{DC} = 0$

$8Q + .8E\theta_B + .4E\theta_B - .30Ed = 0$; ~~$(.15E\theta_B - \frac{.30}{8}Ed = -Q)$~~

$1.2\theta_B - .30d = -\frac{8Q}{E}$ ——————————————— (1)

Case a Let the variation of I in BC be as in Fig. 32a.

$\Delta x \Sigma \frac{1}{I} = 2(2 + \frac{2}{1.2} + \frac{2}{2.4}) = 9$ $\qquad \Delta x \Sigma \frac{x}{I} = 2(1 + \frac{3}{1.2} + \frac{5}{2.4} + \frac{7}{2.4} + \frac{9}{1.2} + 11) = 54$

$\Delta x \Sigma \frac{x^2}{I} = 2(1 + \frac{9}{1.2} + \frac{25}{2.4} + \frac{49}{2.4} + \frac{81}{1.2} + 121) = 455.7$

Denominator $= \int\frac{dx}{I}\int\frac{x^2dx}{I} - \int\frac{xdx}{I}\int\frac{xdx}{I} = 9\times 455.7 - 54^2 = 4100 - 2820$
$= 1280$

Numerator $(\theta_C = \theta_B,\ d = 0) = \theta_B\int\frac{x^2dx}{I} + \theta_C(l\int\frac{xdx}{I} - \int\frac{x^2dx}{I}) - d\int\frac{xdx}{I}$

$= \theta_B\, l\int\frac{xdx}{I} = 12\times 54\,\theta_B = 648\,\theta_B$

$M_{BC} = E\,\frac{648}{1280}\theta_B = .506\,E\theta_B$

$M_{BA} + M_{BC} = 0 \qquad .4E\theta_B - .075Ed + .506E\theta_B = 0$

Or $.906\,\theta_B - .075\,d = 0$

Multiply by 4 $\quad 3.624\,\theta_B - .3d = 0$

Subtract (1) $\quad 1.2\,\theta_B - .3d = -\frac{8Q}{E}$

$$2.424\,\theta_B = \frac{8Q}{E}$$

$$\theta_B = 3.3\,\frac{Q}{E}$$

$M_{BC} = .506\,E\,\theta_B = 1.67\,Q$

Case b Let the variation of I in BC be as shown in Fig. 32b

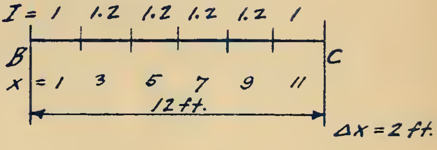


Fig. 32b

$\Delta x \sum \frac{1}{I} = 2\left(2 + \frac{4}{1.2}\right) = 10.67$

$\Delta x \sum \frac{x}{I} = 2\left(\frac{1+11}{1} + \frac{3+5+7+9}{1.2}\right) = 64$

$\Delta x \sum \frac{x^2}{I} = 2\left(\frac{1+121}{1} + \frac{9+25+49+81}{1.2}\right) = 517.5$

Denominator $= 517.5 \times 10.67 - 64^2 = 5515 - 4095 = 1420$

Numerator $(\theta_B = \theta_c;\ d=0) = \ell \int \frac{x\,dx}{I}\,\theta_B = 12 \times 64\,\theta_B = 768\,\theta_B$

$M_{Bc} = E \times \frac{768}{1420}\,\theta_B = .541\,E\,\theta_B$

$M_{BA} + M_{BC} = 0\,;\quad .4\,E\theta_B - .075\,Ed + .541\,E\theta_B = 0$

$.941\,\theta_B - .075\,d = 0$

Multiply by 4 $\quad 3.764\,\theta_B - .3d = 0$

Subtract (1) $\quad 1.2\,\theta_B - .3d = -\frac{8Q}{E}$

$$2.564\,\theta_B = \frac{8Q}{E}$$

$$\theta_B = .311\,\frac{Q}{E}$$

$M_{BC} = .541\,E\,\theta_B = 1.685\,Q$

Case c. Consider I of BC uniform and equal 1.

$M_{BC} = 2E \times \frac{1}{12}\left(2\theta_B + \theta_c - \frac{3d}{12}\right) \qquad \theta_c = \theta_B\,,\ d = 0$

$= \frac{1}{2}\,E\,\theta_B$

$M_{BA} + M_{BC} = 0 \qquad .4E\theta_B + .075Ed + .500E\theta_B = 0$

Multiply by 4 $\qquad 3.600\,\theta_B - .3d = 0$

Subtract (1) $\qquad 1.2\,\theta_B - .3d = -\frac{8Q}{E}$

$$2.40\,\theta_B = \frac{8Q}{E}$$

$$\theta_B = 3.33\frac{Q}{E}$$

$M_{BC} = .50\,E\theta_B = 1.67Q$

Case d $\quad$ Consider I of BC uniform and equal 1.2

$M_{BC} = \frac{2E \times 1.2}{12}(3\theta_B) = .6E\theta_B$

$M_{BA} + M_{BC} = 1.0E\theta_B - .075Ed = 0$

Multiply by 4 $\qquad 4.0\,\theta_B - .3d = 0$

Subtract (1) $\qquad 1.2\,\theta_B - .3d = -\frac{8Q}{E}$

$$2.8\,\theta_B = \frac{8Q}{E}$$

$$\theta_B = 2.86\frac{Q}{E}$$

$M_{BC} = .6E\theta_B = .1715\,Q$

Case e $\quad$ Consider I of BC uniform and equal 2.4

$M_{BC} = \frac{2E \times 2.4}{12}(3\theta_B) = 1.2E\theta_B$

$M_{BA} + M_{BC} = 1.6E\theta_B - .075Ed = 0$

Multiply by 4 $\qquad 6.4\,\theta_B - .3d = 0$

Subtract (1) $\qquad 1.2\,\theta_B - .3d = -\frac{8Q}{E}$

$$5.2\,\theta_B = \frac{8Q}{E}$$

$$\theta_B = 1.54\frac{Q}{E}$$

$M_{BC} = 1.2E\theta_B = .185Q$

Comparison	Case a	b	c	d	e
M_{BC}	.167 Q	.1685Q	.167Q	.1715Q	.185Q
Difference		.0015Q	0	.0045Q	.018Q
%		.9%	0	2.7%	10.8%

Case c. Assume I of full section throughout. Then for I at haunch use the value of I at the point of contraflexure. I is uniform and equals 8.

The work is shown on pages 76 and 77. From the comparison of results it can be seen that using the full section for I throughout gives a result 18% in error; using the part section - only 7%

EXAMPLE 13.

To find the effect on the moment at a joint of the various shortcuts to the modified slope deflection equations. Given a symmetrical frame, Fig 32, with a horizontal load acting in line with the top of the frame. The I of the columns is uniform. The I of the top is about what it would be in the average case.

Case a. Find moment at joint B by means of the modified slope deflection equations. I of top varies as shown in Fig. 32a.

Case b. Use part of section for finding I near point of contraflexure, thus reducing its I 50%, and making it equal to the value of I of the sections. next to the haunch. Solve by the modified slope deflection equations. Fig. 32 b.

Case c. Assume I at haunch to be uniform for the member.

Case d. Assume I of section next to haunch to be uniform the member.

Case e. Use for I of the beam the value of I at the point of contraflexure; thus using the full section as effective for I.

(The work is shown on pages 78, 79 and 80.)

A comparison of the results will show that to assume that I at the haunches and near the haunches predominates in its influence on I of the beam, gives results close to actual.

E. CONCLUSION.

From the previous examples and their discussion, the following conclusions may be drawn:

a. Using the full section as effective for I throughout a beam gives results greatly in error.

b. Neglecting the concrete in tension as effective for I gives results close to actual.

c. For the I of the beam, the sections near the point of contraflexure have little influence, the sections at maximum stress have the most influence.

d. An exact average I of a beam whose sections have a varying I is impossible.

e. Using an I for the beam found by averaging the I's of its most stressed sections gives results quite close to actual.

f. **Better** results are obtained when in finding I of a beam, the average I of the most stressed section is increased somewhat (to allow for the influence of I at the point of contraflexure).

PART II.

APPLICATIONS TO THE DESIGN OF THE ROOSEVELT ROAD VIADUCT.

CHAPTER VII.

GENERAL DESCRIPTION AND HISTORICAL.

The Roosevelt Road Viaduct extends from Wabash Avenue to Canal Street. It consists of three parts: a viaduct from Wabash Avenue to the Chicago River, called the East half; a single leaf bascule bridge across the River providing for 140 feet clear channel and a viaduct from the River to Canal Street called the West half.

The east half of the viaduct consists of a 219 foot filled approach (East Approach) and 1788 feet of viaduct including the State Street crossing which consists of three arch spans. The east half of the viaduct is reached by two crossing streets, Clark and Wells. The North Approach on Clark Street consists of a 602 foot filled part and 234 feet of viaduct; the south Clark Street Approach consists of a 700 foot filled part and 185 feet of viaduct. The Wells Street Approach consists of a ramp abt. 850 feet long. The East half is designed and built by the City of Chicago.

The bascule bridge across the Chicago River presented a complicated problem for the following reasons:

1. Possible diversion of Chicago River Channel.
2. High land damages of railroad property on the East bank of the Chicago River.

The solution of the problem and the various conditions involved in it are explained in an article by Mr. H. E. Young, Engineer of Bridge Design, City of Chicago, in the Engineering World of July 1, 1919. (The bridge is designed by the City and will be built by contract).

The West half of the viaduct consists of 1072 feet of viaduct and a 245 foot filled approach. The West half is reached by Lumber St. Ramp just west of the river. It is designed and built by the Chicago Union Station Co.

This work concerns itself only with the East half of the viaduct.

In accordance with the 12th Street Improvement Ordinance (July 2, 1914) plans were made for a steel viaduct encased in concrete. But when the plans were completed, building operations could not be started; due to the war, steel was scarce and deliveries were uncertain. This led to making a study of a concrete viaduct. The first study brought out the fact that a concrete viaduct could be built for 75% of the cost of the steel viaduct. This and other advantages of the concrete viaduct over the steel, such as:

1. For a steel viaduct shop plans have to be made,
2. Time has to be wasted while the steel is being fabricated,
3. Reinforcing steel could easily be obtained,

led to the adoption of the concrete viaduct.

According to the ordinance, the railroads going underneath the viaduct have to maintain it. The agreement of these railroads to the adopted change had to be obtained. This was difficult; but after pressure was brought on them by the Chicago Plan Commission, an agreement was reached on the following conditions:

1. The number of bents in the concrete viaduct would not exceed the number in the previous steel design.
2. The columns are to be only 2 feet 8 inches wide.

A design for the concrete viaduct was then made. It followed closely the steel design. It, in fact, merely replaced the steel beams by concrete. The analysis was based on the three moment theorem, with assumed values of bending moments at the end supports, that is, at the expansion joint columns.

In February, 1919, a new design was worked out by Mr. R. R. Leffler. The design gives lines that are adapted to a concrete structure. After considerable study by the Chicago Plan Commission, it was adopted as the design for the Concrete Viaduct.

CHAPTER VIII.

FEATURES OF THE DESIGN.

The Twelfth St. Improvement Ordinance calls for maintaining traffic during the construction of the viaduct. To accomplish this, it was thought best to design the viaduct in two longitudinal halves. While the south half is under construction, traffic will be maintained on the present steel viaduct. After the south half is built, the traffic will be shifted to the south half, and the north half built.

The original reinforced concrete design, as shown in Exhibit A, called for longitudinal stringers framing into floor beams, which in turn framed into columns arranged in two longitudinal rows (for each half). The new design, as shown in Exhibit B, combined all these stringers into two longitudinal girders framing into two rows of columns. The slab in the former design, although having short spans, had to be made heavier than required for the stresses in order to get a minimum thickness considered to be good practice for slabs exposed to impact. In the latter design the slab spans a greater distance, and the calculated slab thickness could be used after making allowance in the live load for impact. At intervals of about thirty feet a concrete beam is placed across the girders to stiffen them and make them act together.

In the new design the structure is essentially a slab resting on two girders, with the sidewalk and street car slabs cantilevered, thus giving in effect a three-span continuous and partly fixed slab.

The combining of several stringers into two beams decreased the beam surface and simplified framework. This simplification is of especial importance since the viaduct is over railroad tracks and interference with railroad traffic is thus avoided. Fewer beams also simplify the laying of the reinforcing steel. The smaller quantity of concrete, simpler form work and placing of the steel reduce the cost of the structure.

The new design adapts itself more readily to analysis as a rigid frame, the girders being stiffened as well as supported by the columns. In the old design, the stringers are continuous over their beam supports, but the degree of stiffness of the support is difficult to determine.

From the architectural viewpoint the new design possesses certain advantages. The old design is an imitation of a typical steel design with short spans for slabs and many stringers; it is in fact the same in appearance as a steel structure encased in concrete. The new design is adapted specifically to concrete: it is massive, has few and simple but graceful lines. It eliminates the great number of unsightly beams underneath the structure, substituting for them arched girders blending into the columns; it also does away with brackets supporting cantilever sidewalk beams.

CHAPTER IX.

ADVISABILITY OF USING THE SLOPE-DEFLECTION METHOD.

The usual method of analyzing continuous girders, the three moment theorem, is inadequate for this structure. In reinforced concrete design, methods which do not take account of the rigidity of the connections are not quite accurate and do not give a correct idea of the distribution of tension in concrete, which is a matter of great importance to a structure of this type.

While the three moment theorem does not give an adequate distribution of tensile stresses in the girders, the details can still be taken care of by placing steel in excess of requirements and extending it to a safe distance beyond the probable point of contraflexure. This is not economical, but it is the usual practice.

However, the columns present a different problem. The railroad tracks under the viaduct are spaced close together and square columns of the required area could not be used. The section adopted is 6 feet 7 inches by 2 feet 8 inches, the 2 foot 8 inch dimension being in the direction of the viaduct. Columns of such dimensions are quite rigid, stiffening the girders to a certain extent, and as a result taking bending moments. For a moving load, where only one panel is loaded, the columns supporting that panel get heavy bending moments.

The columns being comparatively shallow (only two feet 8 inches), it is quite important to ascertain the resulting stresses in order not to overstress the columns.

Another important factor is the temperature variation. The expansion joints are located 200 to 240 feet apart. This, in conjunction with the fact that the spans between the expansion joints are by no means symmetrical, gives rise to high temperature stresses. The usual practice is to analyze these stresses by assuming the amount of contraction or expansion to be equal to the deflection of one end of the column which is held rigid at its other end, and by figuring the stresses which would cause such a deflection. This method is primitive and cannot inspire confidence into the results obtained by it. Due to their shallowness, the columns, are apt to become highly stressed from moments acting on them. The necessity of a rational analysis, the results of which would give a correct idea of the stresses in the columns is obvious.

The slope deflection method gives a rational analysis of the structure. It gives the extent of tensile stresses in the girder due to moving live loads, and also the moments in the columns due to live loads and temperature variations.

CALCULATIONS FOR TYPICAL SECTION.

The section chosen to illustrate the application of the slope deflection method is one just to the east of Clark Street between bents 21 and 25. Its general dimensions are shown in Figure 33 Page 96 .

Points A-A in Figure 33 are connected by a circular curve. The ordinate to this curve from the middle of the chord connecting A-A is given as the camber distance. This camber is made partly to take up deflection but mostly to counteract an optical illusion, which would make line A-A appear sagging when viewed from a distance. The camber is 1/2 inch for every 12 feet of line A-A.

A. PRELIMINARY ASSUMPTIONS.

1. It is well known that all methods of analyzing statically indeterminate structures depend on the elastic properties of the frame and require a knowledge of the sections and the distribution of material in the sections. In other words the moment of inertia of the sections of the various members of the frame must be known.

In analyzing statically indeterminate structures, it is usual to assume the sections, based upon some approximate computations, calculate the moments of inertia and then use these values in the analysis. With the aid of the stresses found by analysis, the assumed sections are reviewed and where necessary, changed.

These changes mean a change in the moments of inertia. But usually such changes are relatively small and do not affect the stresses seriously. When the approximate distribution of moments of inertia throughout a frame is known, it takes a comparatively large change in the moment of inertia to make an appreciable change in the stresses. * With practice it becomes possible to choose the sections by approximate design so that no material changes will be required.

In the present example the sections were chosen in the following way: the end supports of the girders were assumed to take a bending moment of $\frac{wl^2}{90}$, the interior columns were assumed not to influence the rotation of the girders resting on them. The three moment theorem was applied to obtain the bending moments and reactions on the girders. From these and from considerations of necessary clearance for railroads under the viaduct, the sections were chosen; then the reinforcing steel calculated. The sections and the steel thus found served as a basis for obtaining the moments of inertia of the girders.

*For effect of changing moment of inertia, see University of Illinois Eng. Exp. Sta., Bulletin #80, page 27; Bulletin #107, pages 48 to 54 and Hool and Johnson's "Concrete Engineers' Handbook," pages 421 to 424.

The bending moments in the columns are caused mostly by traction and temperature variations. A preliminary investigation by the slope deflection method was made assuming for simplicity a symmetrical layout. The bending moments thus obtained, together with the reaction figured before, served to indicate the amount of steel required in the columns.

2. As is seen from Fig 33 , page 96 the columns bear on caisson caps placed at least nine feet below the surface of the ground. The weight of the ground above the cap and the driving of the caissons consolidate the ground around the caissons; it can be safely assumed that this solid ground will prevent, to a large extent, any movement of the top of caisson due to bending in the column. Since there is reinforcing steel well down in the caissons, extending up and dowelling into the columns, it may be assumed that the columns are fixed at the caissons.

According to page 628, article 8-a of Hool and Johnson's "Concrete Engineers' Handbook", it is safest to consider the column ends as fixed for vertical loads, and as hinged for tractive forces. In this case the stresses caused by traction are small in comparison with those caused by the vertical loads and hence the stresses obtained will be on the safe side if the column ends are assumed as fixed.

3. Figure 33 , page 96 shows that the top of the girder is at a small grade. Also the underside of the girder is curved and at a different grade from the top. In the analysis, these various grades were neglected.*

4. In trying to replace each member of the frame by a line for the skeleton diagram from which the analysis is made, another difficulty is encountered. Such a line should be the neutral axis of the member. But the neutral axis of the girder or a column subjected to combined bending and direct stresses due to moving loads, cannot be a straight line. The curve which represents the neutral axis changes its position and form as the load moves from point to point. As compared with the span of the girders and the height of the columns these variations of the neutral axis from a straight line are negligible. Hence the center lines of the columns and the lines joining the mid-depths of the girders at points A - A will be considered as the lines of action for the columns and girders respectively.

B. LOADS.

1. The dead load, consisting of the weight of the slabs, railing, pavement, cross girders and girders, is considered as uniformly distributed over the girder span. Calculations not shown here, gave a value close to 13000 pounds. Hence this was adopted as the dead load per linear foot of girder.

---

* Compare in Hool and Johnson's "Concrete Engineers' Handbook" Type II, Case 1 on page 428 assuming a small grade of the top with Type III, Case 2 on page 789.

2. The girder under consideration carries a sidewalk and roadway slab. The live load on these slabs is given in the specifications used in designing.* The impact allowance was also made according to these specifications. The resulting load was considered as uniformly distributed and amounted to 4600 pounds per lineal foot of girder.

3. For the calculation of temperature stresses a basic temperature of 65 degrees Fahrenheit was assumed and the effects of a decrease to -15 degrees Fahrenheit and an increase to 105 degrees Fahrenheit were calculated. The expansion joint is made one inch at 65 degrees Fahrenheit, with a proportional variation when construction is under another temperature.

4. The effect of longitudinal forces, or traction, was figured according to the specifications mentioned before.

C. DETAILS.

The details required for the calculations of the moments of inertia are shown in Figure 33, page 96 and in the accompanying blueprints. Thus the general dimensions of the girders and the columns are shown in Figure 33, page 96, the width of girders and columns is shown on drawing #7210; the reinforcing steel in the girders is shown on drawings #7230 and #7231, and for the columns is shown on drawing #7253. The detail of the knee brace is shown on drawing #7251.

*"Specifications Governing the Construction, Repairs and Rebuilding of Viaducts over Railroad Tracks" issued by the Bureau of Engineering, Department of Public Works, City of Chicago, January, 1917.

D. Moments of Inertia and Coefficients of Rigidity.

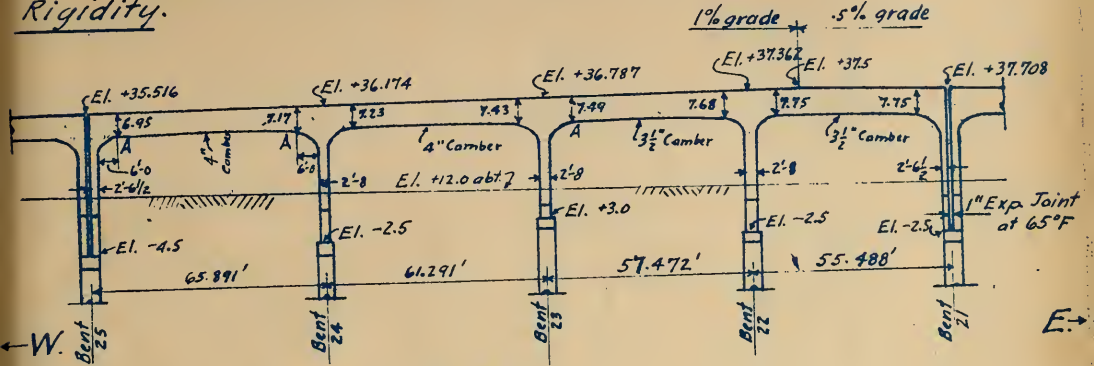
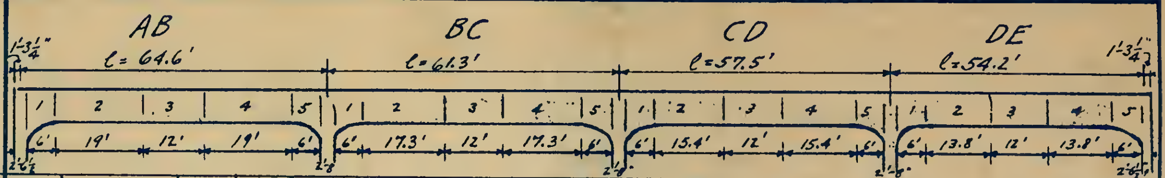


Fig. 33

General Elevation of Frame.

Calculations for coefficients of rigidity for the girders and columns are shown in Tables 4 and 5 respectively. Each girder as is seen from Table 4 was divided into five parts; the moment of inertia of each part was found and from this the reciprocal of the coefficient of rigidity. These were summed up and inverted giving the coefficient of rigidity of the girder.

The interior columns have either compression over the whole section or very little tension. For these columns the moment of inertia was figured from the full section. The exterior, or expansion joint, columns have high tensile stresses in the top and bottom sections. These stresses will cause cracks on the tension side of the section, so that only part of the section will be effective for moment



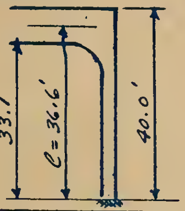
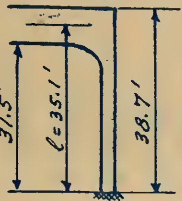
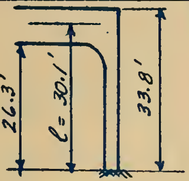
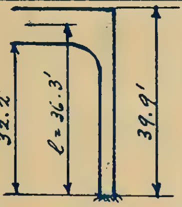
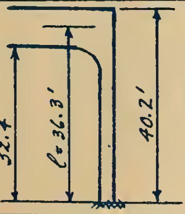
Girder	AB $l = 64.6'$					BC $l = 61.3'$					CD $l = 57.5'$					DE $l = 54.2'$				
Section	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
Steel in Top	11-3/4" 4-1/4"	11-3/4"	11-3/4"	11-3/4"	29-1/4" 10-3/4"	29-1/4" 10-3/4"	10-1/4" 10-3/4"	14-3/4"	4-1/4" 11-3/4"	20-1/4" 7-3/4"	20-1/4" 7-3/4"	6-1/4" 9-3/4"	14-3/4"	6-1/4" 10-3/4"	22-1/4" 7-3/4"	22-1/4"	6-1/4" 10-3/4"	14-3/4"	14-3/4"	14-3/4"
Steel in bottom	25-1/4"	46-1/4"	51-1/4"	30-1/4"	16-1/4"	6-3/4" 10-1/4"	16-1/4"	27-1/4"	14-1/4"	6-3/4" 9-1/4"	12-1/4" 3-3/4"	17-1/4"	23-1/4"	16-1/4"	6-3/4" 9-1/4"	7-3/4"	18-1/4"	33-1/4"	30-1/4"	18-1/4"
Average I of Section in ft.^{4}	124.0	170.7	146.5	114.0	238.3	212.5	103.1	111.0	86.6	161.6	167.1	100.0	111.6	100.8	186.6	194.8	117.8	148.3	143.3	118.8
$\frac{\Delta l}{I}$	.0484	.111	.0818	.166	.0252	.0283	.168	.1082	.200	.0371	.036	.154	.1035	.153	.0321	.0308	.117	.081	.096	.0505
$\frac{1}{K} = \Sigma \frac{\Delta l}{I}$	.4324					.5415					.4785					.3753				
K	2.3					1.85					2.09 (Used 2.10)					2.67 (Used 2.65)				
$\frac{6K}{l}$	.214					.181					.219					.294				

TABLE 4

Coefficients of Rigidity of Girders.

TABLE 5

Coefficients of Rigidity of Columns

Column	Depth of Col.	Steel in Column	ΔC	Average I of portion in ft^4	$\frac{\Delta l}{I}$	$\frac{1}{K}=\sum\frac{\Delta l}{I}$	K	$\frac{6K}{l}$
AF 	2'-6$\frac{1}{2}$	18-1$\frac{1}{4}$"	8.1 15.0 7.5	5.21 11.51 6.01	1.556 1.302 1.249	4.107	.25	.041
BG 	2'-8	26-1$\frac{1}{4}$"	29.1	14.64			.45	.077
CH 	2'-8	18-1"	23.9	12.0			.45	.090
DJ 	2'-8	26-1$\frac{1}{4}$"	30.0	14.64			.45	.074
EK 	2'-6$\frac{1}{2}$	18-1$\frac{1}{4}$"	11.6 18.2	5.70 11.51	2.035 1.581	3.616	.25	.041

of inertia. Each column has a point of contraflexure somewhere near the middle of its length. For a certain distance up and down of the point of contraflexure the tensile stresses will not be high enough to cause cracks in the concrete. For that portion of the length of the column the full section is figured for the moment of inertia. In the case of column AF fifteen feet were figured for the full section; 8.1 feet, with the neutral axis placed 1.3 feet from the compressive face and 7.5 feet, with neutral axis 1.7 feet from the compressive face of the section. For column EK, 18.2 feet for the full section and 11.6 feet with neutral axis 1.6 feet from the compressive face.

E. EQUATIONS FOR MOMENTS.

Figure 34, Page 100 shows the skeleton diagram of the frame used in the analysis. The dotted lines show the deformations of the members due to contraction. The deflections of the various members for this case are worked out in Table 6 . The coefficient of contraction and expansion is taken as .0000066 foot per foot per degree F. Note that the coefficients of t are multiplied by 100000. The results are later divided by 100000 to leave the moments and reactions unchanged.

The equations for moments worked out in Chapter III are:

$$M_{AB} = \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3d}{l}\right) + C_{AB}$$

$$M_{BA} = \frac{2EI}{l}\left(\theta_A + 2\theta_B - \frac{3d}{l}\right) + C_{BA}$$

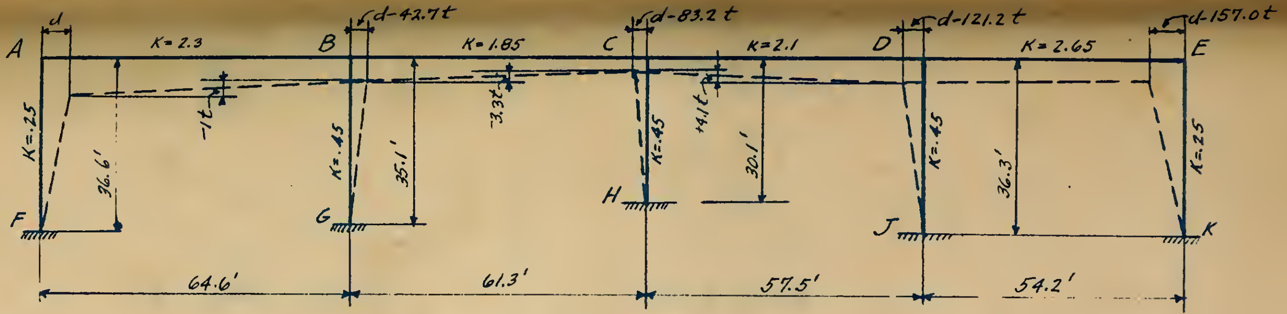


FIG. 34

Member	Length	Contraction$_t$	Deflection($t \times 100000$)	$\frac{6K}{\ell} \times$ Deflection
AB	64.6	.000427 t	$-(24.2-23.2)t = -1t$	$-.214t$
BC	61.3	.000405 t	$-(23.2-19.9)t = -3.3t$	$-.598t$
CD	57.5	.000380 t	$-(19.9-24)t = +4.1t$	$+.898t$
DE	54.2	.000358 t	0	0
AF	36.6	.000242 t	d	$.041d$
BG	35.1	.000232 t	$d-42.7t$	$.077d-3.280t$
CH	30.1	.000199 t	$d-83.2t$	$.090d-7.460t$
DJ	36.3	.000240t	$d-121.2t$	$.074d-9.010t$
EK	36.3	.000240t	$d-157.0t$	$.041d-6.480t$

TABLE 6

Deformations Due to Contraction

For ease of handling in table 7, the form of the equations is changed to: -

$$M_{AB} = 4KE\theta_A + 2KE\theta_B - \frac{6K}{e}Ed + C_{AB}$$

$$M_{BA} = 2KE\theta_A + 4KE\theta_B - \frac{6K}{e}Ed + C_{BA}$$

Table 7 contains expressions for moments at the ends of each member of the frame shown in Figure 34, page 100. Eight cases are considered: -

1. Dead load
2. Live load on girder AB
3. " " " " BC
4. " " " " CD
5. " " " " DE
6. Traction
7. Contraction
8. Expansion

Bending moments caused by expansion are equal and opposite in sign to those caused by contraction.

The moments and reactions at the joints are at first worked out in terms of coefficients. The coefficient for vertical loads in cases 1 to 5 is w pounds per foot; for traction in case 6 is Q pounds and for temperature variations in cases 7 and 8 is t degrees.

TABLE 7. MOMENT EQUATIONS

		VARIABLE TERMS						CONSTANT TERMS						
								Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
		$E\theta_A$	$E\theta_B$	$E\theta_C$	$E\theta_D$	$E\theta_E$	Ed	W	W	W	W	W	Q	Et/100000
M_{AF}	=	+1.0					−.041							
M_{AB}	=	+9.2	+4.6					−348	−348					+.214
M_{BA}	=	+4.6	+9.2					+348	+348					+.214
M_{BG}	=		+1.8				−.077							+3.280
M_{BC}	=		+7.4	+3.7				−313		−313				+.598
M_{CB}	=		+3.7	+7.4				+313		+313				+.598
M_{CH}	=			+1.8			−.090							+7.460
M_{CD}	=			+8.4	+4.2			−275			−275			−.898
M_{DC}	=			+4.2	+8.4			+275			+275			−.898
M_{DJ}	=				+1.8		−.074							+9.010
M_{DE}	=				+10.6	+5.3		−245				−245		
M_{ED}	=				+5.3	+10.6		+245				+245		
M_{EK}	=					+1.0	−.041							+6.480
M_{FA}	=	+.5					−.041							
M_{GB}	=		+.9				−.077							+3.280
M_{HC}	=			+.9			−.090							+7.460
M_{JD}	=				+.9		−.074							+9.010
M_{KE}	=					+.5	−.041							+6.480

The numerical values of the coefficients are as follows: -

Case 1 , $W = 13000$ pounds
Case 2 to 5, $W = 4600$ pounds
Case 6 , $Q = 20000$ pounds
Case 7 , $t = 80$ degrees
Case 8 , $t = 40$ degrees.

These values were discussed in section "B" of this chapter.

The loading factors used in the expressions for moments at the joints were assumed as for girders with constant moment of inertia. Hence the loading factor.

$$C = \frac{w\ell^2}{12}$$

and is the same for both ends of the member.

For girder AB, $C = w \times \frac{64.6^2}{12} = 348\ W$

" " BC, $C = w \times \frac{61.3^2}{12} = 313\ W$

" " CD, $C = w \times \frac{57.5^2}{12} = 275\ W$

" " DE $C = w \times \frac{54.2^2}{12} = 245\ W$

F. GENERAL EQUATIONS FOR THE FRAME.

The general equations in Table 8 are obtained as follows. Consider joint B. Three moments are acting at that joint: M_{BA} , M_{BG} and M_{BC} . For equilibrium the sum of these moments must equal zero. Summing up the moments at the joints A, B, C, D and E gives five equations.

TABLE 8. GENERAL EQUATIONS

No.	UNKNOWN TERMS						KNOWN TERMS								
	$E\theta_A$	$E\theta_B$	$E\theta_C$	$E\theta_D$	$E\theta_E$	Ed	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7		
							W	W	W	W	W	Q	t		
1	+ 10.2	+ 4.6				− .041	− 348	− 348					+ .214	=	0
2	+ 4.6	+ 18.4	+ 3.7			− .077	+ 35	+ 348	− 313				+ 4.092	=	0
3		+ 3.7	+ 17.6	+ 4.2		− .090	+ 38		+ 313	− 275			+ 7.160	=	0
4			+ 4.2	+ 20.8	+ 5.3	− .074	+ 30			+ 275	− 245		+ 8.112	=	0
5				+ 5.3	+ 11.6	− .041	+ 245				+ 245		+ 6.480	=	0
6	+ .041	+ .077	+ .090	+ .074	+ .041	− .0189						− 1	+ 1.537	=	0

TABLE 9. CALCULATIONS FOR EQUATION 6

		$E\theta_A$	$E\theta_B$	$E\theta_C$	$E\theta_D$	$E\theta_E$	Ed	$Et/100000$
$\frac{M_{AF}+M_{FA}}{\ell}$	=	$\frac{1.5}{36.6} = .041$					$\frac{.082}{36.6} = -.00224$	
$\frac{M_{BG}+M_{GB}}{\ell}$	=		$\frac{2.7}{35.1} = .077$				$\frac{.154}{35.1} = -.00438$	$+\frac{6.56}{35.1} = +.1868$
$\frac{M_{CH}+M_{HC}}{\ell}$	=			$\frac{2.7}{30.1} = .090$			$\frac{.180}{30.1} = -.00598$	$+\frac{14.92}{30.1} = +.4960$
$\frac{M_{DJ}+M_{JD}}{\ell}$	=				$\frac{2.7}{36.3} = .074$		$\frac{.148}{36.3} = -.00408$	$+\frac{18.02}{36.3} = +.4970$
$\frac{M_{EK}+M_{KE}}{\ell}$	=					$\frac{1.5}{36.3} = .041$	$\frac{.082}{36.3} = -.00226$	$+\frac{12.96}{36.3} = +.3570$
Σ	=	+.041	+.077	+.090	+.074	+.041	−.01894	+1.5368

Consider the columns as free members.. The moments at the top and bottom of the columns cause horizontal reactions. Chapter IV shows how to derive these reactions when the moments are known. For equilibrium the sum of the horizontal reactions at the foot of the columns must equal zero in cases 1, 2, 3, 4, 5, 7 and 8. In case 6 that sum must equal Q. Equation 6 of Table 8 expresses this condition of equilibrium. Calculations for the horizontal reactions are shown in Table. 9 .

Table 10 works out the values of the unknowns in the general equations in terms of W, Q and t. These values are substituted in the expressions for moments at the joints, thus giving moment coefficients. See Table 11 . Note that in cases 1 to 6 the modulus of elasticity, E, cancels out; but in cases 7 and 8 it does not. Since all dimensions of the frame are given in feet and the moment of inertia in $ft.^4$, E must be expressed in pounds per square foot. Hence E = 288,000,000.

The numerical values of the moment coefficients are substituted in Table 12 . From the moments the reaction coefficients are worked out in Table 13 . The numerical values of the reaction coefficients are substituted in Table 14 .

G. COLUMN STRESSES.

With the bending moments and the reactions on the columns known, the stresses are figured out in Tables 15 and 16 .

TABLE 10. SOLUTION OF GENERAL EQUATIONS

1.

	Operation			UNKNOWN TERMS					KNOWN TERMS							No. of Equation
				θ_B	θ_C	θ_D	θ_E	d	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	
									W	W	W	W	W	Q	$Et/10000$	
Eliminate θ_A	From 1.	$10.2\,\theta_A$	=	−4.6				+.041	+348	+348					−.214	
	∴	θ_A	=	−.451				+.00402	+34.1	+34.1					−.02095	1a
	∴	$.041\,\theta_A$	=	−.0185				+.0001648	+1.40	+1.40					−.00086	1b
	∴	$4.6\,\theta_A$	=	−2.075				+.0185	+156.9	+156.9					−.0965	1c
	Subst. 1b in 6	0	=	+.0585	+.090	+.074	.041	−.0187352	+1.40	+1.40				−1	+1.53614	6a
	Subst. 1c in 2	0	=	+16.325	+3.7			−.0585	+191.9	+504.9	−313				+3.9955	2a
Eliminate θ_E	From 5	$11.6\,\theta_E$	=			−5.3		+.041	−245				−245		−6.480	
	∴	θ_E	=			−.457		+.00353	−21.1				−21.1		−.5580	5a
	∴	$.041\,\theta_E$	=			−.0187		+.0001448	−.865				−.865		−.0229	5b
	∴	$5.3\,\theta_E$	=			−2.420		+.0187	−111.8				−111.8		−2.960	5c
	Subst 5b in 6a	0	=	+.0585	+.090	+.0553		−.0185904	+.535	+1.40			−.865	−1	+1.51324	6b
	Subst. 5c in 4	0	=		+4.2	+18.38		−.0553	−81.8			+275	−356.8		+5.152	4a
Eliminate θ_B and θ_D	From 2a	$16.325\,\theta_B$	=		−3.7			+.0585	−191.9	−504.9	+313				−3.9955	
	∴	θ_B	=		−.227			+.00359	−11.76	−30.9	+19.18				−.245	2b
	∴	$.0585\,\theta_B$	=		−.0133			+.0002095	−.69	−1.81	+1.12				−.01432	2c
	∴	$3.7\,\theta_B$	=		−.838			+.0133	−43.5	−114.3	+70.9				−.905	2d
	From 4a	$18.38\,\theta_D$	=		−4.2			+.0553	+81.8			−275	+356.8		−5.152	
	∴	θ_D	=		−.228			+.00301	+4.45			−14.95	+19.53		−.2805	4b
	∴	$.0553\,\theta_D$	=		−.0126			+.0001664	+.246			−.827	+1.08		−.0155	4c
	∴	$4.2\,\theta_D$	=		−.959			+.0126	+18.7			−62.8	+82.0		−1.178	4d
	Subst. 2c & 4c in 6b	0	=		+.0641			−.0182145	+.091	−.41	+1.12	−.827	+.215	−1	+1.48342	6c
	Subst. 2d & 4d in 3	0	=		+15.803			−.0641	+13.2	−114.3	+383.9	−337.8	+82		+5.077	3a

TABLE 10. (concluded)

2.

				UNKNOWN TERMS					KNOWN TERMS							No. of Equation
									Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	
	Operation			θ_B	θ_C	θ_D	θ_E	d	W	W	W	W	W	Q	$Et/100000$	
Elimin. θ_C	From 3a	$15.803\ \theta_C$	=					+.0641	-13.2	+114.3	-383.9	+337.8	-82		-5.077	
	∴	θ_C	=					+.00406	-.835	+7.23	-24.3	+21.4	-5.18		-.321	3b
	∴	$.0641\ \theta_C$	=					+.0002595	-.0535	+.463	-1.556	+1.37	-.333		-.0206	3c
	Subst. 3c in 6c	0	=					-.017955	+.0375	+.053	-.436	+.543	-.118	-1	+1.46282	6d
Solution	From 6d	Ed	=						+2.09	+2.95	-24.3	+30.25	-6.57	-55.7	+81.52	S1
	Subst. S1 in 3b	$E\theta_C$	=						-.827	+ 7.24	-24.4	+21.52	-5.21	-.226	+.010	S2
	Subst. S1 & S2 in 4b	$E\theta_d$	=						+4.65	-1.64	+5.50	-19.77	+20.70	-.116	-.0370	S3
	Subst. S1 & S2 in 2b	$E\theta_B$	=						-11.56	-32.53	+24.62	-4.77	+1.16	-.149	+.0455	S4
	Subst. S1 & S4 in 1a	$E\theta_A$	=						+39.32	+48.77	-11.2	+2.27	-.54	-.156	+.28605	S5
	Subst. S1 & S3 in 5a	$E\theta_E$	=						-23.19	+.76	-2.60	+9.14	-30.62	-.1435	-.2536	S6

TABLE 11. MOMENT COEFFICIENTS

1.

		Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
		Dead Load	LL. on AB	LL. on BC	LL. on CD	LL. on DE	Traction	Contraction
		w	w	w	w	w	Q	$Et/100000$
$E\theta_A$	=	+ 39.3	+ 48.8	− 11.2	+ 2.3	− .5	− .16	+ .286
Ed	=	− .1	− .1	+ 1.0	− 1.2	− .3	+ 2.28	− 3.338
M_{AF}	=	+ 39.2	+ 48.7	− 10.2	+ 1.1	− .8	+ 2.12	− 3.052
$E\theta_A$	=	+ 361.8	+ 448.2	− 103.2	+ 20.9	− 5.0	− 1.44	+ 2.635
$E\theta_B$	=	− 53.2	− 149.7	+ 113.4	− 21.9	+ 5.3	− .69	+ .209
w_1 or t	=	− 348.0	− 348.0					+ .214
M_{AB}	=	− 39.4	− 49.5	+ 10.3	− 1.0	+ .3	− 2.13	+ 3.058
$E\theta_A$	=	+ 180.9	+ 224.1	− 51.6	+ 10.5	− 2.5	− .72	+ 1.318
$E\theta_B$	=	− 106.3	− 299.4	+ 226.8	− 43.8	+ 10.7	− 1.37	+ .418
w_1 or t	=	+ 348.0	+ 348.0					+ .214
M_{BA}	=	+ 422.6	− 272.7	+ 175.2	− 33.3	+ 8.2	− 2.09	+ 1.950
$E\theta_B$	=	− 20.8	− 58.6	+ 44.3	− 8.6	+ 2.1	− .27	+ .082
Ed t	=	− .2	− .2	+ 1.9	− 2.3	+ .5	+ 4.29	− 6.280 + 3.280
M_{BE}	=	− 21.0	− 58.8	+ 46.2	− 10.9	+ 2.6	+ 4.02	− 2.918
$E\theta_B$	=	− 85.5	− 240.6	+ 182.4	− 35.3	+ 8.6	− 1.10	+ .337
$E\theta_C$	=	− 3.1	+ 26.8	− 90.2	+ 79.5	− 19.3	− .84	+ .037
w_1 or t	=	− 313.0		− 313.0				+ .598
M_{BC}	=	− 401.6	− 213.8	− 220.8	+ 44.2	− 10.7	− 1.94	+ .972
$E\theta_B$	=	− 42.8	− 120.3	+ 91.2	− 17.7	+ 4.3	− .55	+ .168
$E\theta_C$	=	− 6.1	+ 53.6	− 180.4	+ 159.1	− 38.5	− 1.67	+ .074
w_1 or t	=	+ 313.0		+ 313.0				+ .598
M_{CB}	=	+ 264.1	− 66.7	+ 223.8	+ 141.4	− 34.2	− 2.22	+ .840

TABLE II. (continued)

		Case 1		Case 2		Case 3		Case 4		Case 5		Case 6		Case 7	
		Dead Load		LL. on AB		LL. on BC		LL. on CD		LL. on DE		Traction		Contraction	
		w		w		w		w		w		Q		$Et/100000$	
$E\theta_C$	=	−	1.5	+	13.0	−	43.9	+	38.7	−	9.4	−	.41	+	.018
$Ed \quad t$	=	−	.2	−	.3	+	2.2	−	2.7	−	.6	+	5.01	− +	7.340 7.460
M_{CH}	=	−	1.7	+	12.7	−	41.7	+	36.0	−	10.0	+	4.60	+	.138
$E\theta_C$	=	−	7.0	+	60.8	−	205.2	+	80.8	−	43.8	−	1.90	+	.084
$E\theta_d$	=	+	19.5	−	6.9	+	23.1	−	83.1	+	87.0	−	.49	−	.155
w_1 or t	=	−	275.0					−	275.0					−	.898
M_{CD}	=	−	262.5	+	53.9	−	182.1	−	177.3	+	43.2	−	2.39	−	.969
$E\theta_C$	=	−	3.5	+	30.4	−	102.6	+	90.4	−	21.9	−	.95	+	.042
$E\theta_d$	=	+	39.1	−	13.8	+	46.2	−	166.2	+	174.0	−	.97	−	.311
w_1 or t	=	+	275.0					+	275.0					−	.898
M_{DC}	=	+	310.6	+	16.6	−	56.4	+	199.2	+	152.1	−	1.92	−	1.167
$E\theta_D$	=	+	8.4	−	3.0	+	9.9	−	35.6	+	37.2	−	.21	−	.067
$Ed \quad t$	=	−	.2	−	.2	+	1.8	−	2.2	−	.5	+	4.12	− +	6.035 9.010
M_{DJ}	=	+	8.2	−	3.2	+	11.7	−	37.8	+	36.7	+	3.91	+	2.908
$E\theta_D$	=	+	49.3	−	17.4	+	58.3	−	209.5	+	219.6	−	1.23	−	.392
$E\theta_E$	=	−	123.0	+	4.0	−	13.8	+	48.4	−	162.3	−	.76	−	1.344
w_1 or t	=	−	245.0							−	245.0				
M_{DE}	=	−	318.7	−	13.4	+	44.5	−	161.1	−	187.7	−	1.99	−	1.736
$E\theta_D$	=	+	24.6	−	8.7	+	29.1	−	104.8	+	109.8	−	.61	−	.196
$E\theta_E$	=	−	246.0	+	8.1	−	27.6	+	96.8	−	324.6	−	1.52	−	2.688
w_1 or t	=	+	245.0							+	245.0				
M_{ED}	=	+	23.6	−	.6	+	1.5	−	8.0	+	30.2	−	2.13	−	2.884

TABLE II. (concluded)

3.

		Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
		Dead Load	LL. on AB	LL. on BC	LL. on CD	LL. on DE	Traction	Contraction
		w	w	w	w	w	Q	Et/100000
$E\theta_E$	=	− 23.2	+ .8	− 2.6	+ 9.1	− 30.6	− .14	− .254
$Ed\ \ t$	=	− .1	− .1	+ 1.0	− 1.2	− .3	+ 2.28	− 3.338 + 6.480
M_{EK}	=	− 23.3	+ .7	− 1.6	+ 7.9	− 30.9	+ 2.14	+ 2.888
$E\theta_A$	=	+ 19.6	+ 24.4	− 5.6	+ 1.1	− .2	− .08	+ .143
Ed	=	+ .1	+ .1	+ 1.0	− 1.2	− .3	+ 2.28	− 3.338
M_{FA}	=	+ 19.7	+ 24.5	− 4.6	− .1	− .5	+ 2.20	− 3.185
$E\theta_B$	=	− 10.4	− 29.3	+ 22.3	− 4.3	+ 1.1	− .13	+ .041
$Ed\ \ t$	=	− .2	− .2	+ 1.9	− 2.3	+ .5	+ 4.29	− 6.280 + 3.280
M_{GB}	=	− 10.6	− 29.5	+ 24.2	− 6.6	+ 1.6	+ 4.16	− 2.959
$E\theta_C$	=	− .8	+ 6.5	− 21.9	+ 19.3	− 4.7	− .20	+ .009
$Ed\ \ t$	=	− .2	− .3	+ 2.2	− 2.7	− .6	+ 5.01	− 7.340 + 7.460
M_{HC}	=	− 1.0	+ 6.2	− 19.7	+ 16.6	− 5.3	+ 4.81	+ .129
$E\theta_D$	=	+ 4.2	− 1.5	+ 5.0	− 17.8	+ 18.6	− .10	− .033
$Ed\ \ t$	=	− .2	− .2	+ 1.8	− 2.2	− .5	+ 4.12	− 6.035 + 9.010
M_{JD}	=	+ 4.0	− 1.7	+ 6.8	− 20.0	+ 18.1	+ 4.02	+ 2.942
$E\theta_E$	=	− 11.6	+ .4	− 1.3	+ 4.6	− 15.3	− .07	− .127
$Ed\ \ t$	=	− .1	− .1	+ 1.0	− 1.2	− .3	+ 2.28	− 3.338 + 6.480
M_{KE}	=	− 11.7	+ .3	− .3	+ 3.4	− 15.6	+ 2.21	+ 3.015

TABLE 12. MOMENTS AT JOINTS

1.

		Case 1 W = 13000 #	Case 2 W = 4600 #	Case 3 W = 4600 #	Case 4 W = 4600 #	Case 5 W = 4600 #	Case 6 Q = 20000 #	Case 7 Et/100000 = 230400	Case 8 Et/100000 = 115200
M_{AF}	=			- 47 000		- 4 000	- 42 000	- 704 000	
		+ 510 000	+ 224 000		+ 5 000		+ 42 000		+ 352 000
M_{AB}	=	- 512 000	- 228 000		- 5 000		- 43 000		- 352 000
				+ 47 000		+ 1 000	+ 43 000	+ 705 000	
M_{BA}	=				- 153 000		- 42 000		- 225 000
		+ 5 497 000	+ 1 254 000	+ 806 000		+ 38 000	+ 42 000	+ 449 000	
M_{BG}	=	- 273 000	- 270 000		- 50 000		- 80 000	- 673 000	
				+ 213 000		+ 12 000	+ 80 000		+ 336 000
M_{BC}	=	- 5 224 000	- 984 000	- 1 017 000		- 49 000	- 39 000		- 112 000
					+ 203 000		+ 39 000	+ 224 000	
M_{CB}	=		- 307 000			- 157 000	- 44 000		- 97 000
		+ 3 436 000		+ 1 029 000	+ 651 000		+ 44 000	+ 194 000	
M_{CH}	=	- 22 000		- 192 000		- 46 000	- 92 000		- 16 000
			+ 58 000		+ 166 000		+ 92 000	+ 32 000	
M_{CD}	=	- 3 414 000		- 838 000	- 816 000		- 48 000	- 223 000	
			+ 249 000			+ 199 000	+ 48 000		+ 112 000
M_{DC}	=			- 259 000			- 38 000	- 269 000	
		+ 4 038 000	+ 76 000		+ 916 000	+ 700 000	+ 38 000		+ 134 000
M_{DJ}	=		- 15 000		- 174 000		- 78 000		- 344 000
		+ 107 000		+ 54 000		+ 169 000	+ 78 000	+ 688 000	
M_{DE}	=	- 4 144 000	- 61 000		- 741 000	- 864 000	- 40 000	- 400 000	
				+ 205 000			+ 40 000		+ 200 000
M_{ED}	=		- 3 000		- 37 000		- 43 000	- 665 000	
		+ 314 000		+ 7 000		+ 139 000	+ 43 000		+ 332 000

TABLE 12. (concluded) 2.

		Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8
		w = 13000 #	w = 4600 #	w = 4600 #	W = 4600 #	w = 4600 #	Q = 20000 #	t = 80°	t = 40°
M_{EK}	=	− 306000		− 7000		− 142 000	− 43 000		− 334 000
			+ 3000		+ 36 000		+ 43 000	+ 668 000	
M_{FA}	=			− 21000	0	− 2000	− 44000	− 735000	
		+ 268 000	+ 113 000				+ 44 000		+ 367000
M_{GB}	=	− 109000	− 136000		− 30 000		− 83 000	− 682 000	
				+ 111000		+ 7000	+ 83 000		+ 341000
M_{HC}	=			− 91 000		− 24 000	− 96000		− 15000
		+ 22 000	+ 29 000		+ 76000		+ 96000	+ 30000	
M_{JD}	=		− 8000		− 92 000		− 80 000		− 339 000
		+ 82000		+ 31000		+ 83000	+ 80000	+ 678 000	
M_{KE}	=	− 138000		− 1000		− 72 000	− 44000		− 348000
			+ 1000		+ 16 000		+ 44 000	+ 695 000	

TABLE 13. REACTION COEFFICIENTS

		Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8
		W	W	W	W	W	Q	Et/100000	Et/100000
$-M_{AB}-M_{BA}$	=	− 383.2	− 223.2	− 185.5	+34.3	−8.5	+ 4.22	− 5.008	
· /l	=	− 5.9	−3.5	−3.0	+ .5	−.1	+ .07	−.08	
1/2 wl	=	+ 32.30	+ 32.3						
V_{AB}	=	+ 26.4	+28.8	−3.0	+.5	−.1	+.07	−.08	+.08
	=	+ 5.9	+ 3.5	+3.0	−.5	+.1	−.07	+.08	
1/2 wl	=	+ 32.30	+ 32.3						
V_{BA}	=	+ 38.2	+ 35.8	+3.0	−.5	+.1	−.07	+.08	−.08
$-M_{BC}-M_{CB}$	=	+ 137.5	+ 280.5	−3.0	− 185.6	+ 44.9	+ 4.16	− 1.812	
" /l	=	+ 2.2	+ 4.6	−.05	− 3.0	+ .7	+.07	− .03	
1/2 wl	=	+ 30.65		+ 30.65					
V_{BC}	=	+32.85	+4.6	+ 30.6	−3.0	+.7	+.07	−.03	+.03
	=	− 2.2	− 4.6	+ .05	+3.0	−.7	−.07	+.03	
1/2 wl	=	+ 30.65		+30.65					
V_{CB}	=	+ 28.45	−4.6	+30.7	+3.0	−.7	−.07	+.03	− .03
$-M_{CD}-M_{DC}$	=	− 48.1	− 70.5	+ 238.5	− 21.9	− 195.3	+ 4.31	+ 2.136	
" /l	=	−.8	−1.2	+4.1	−.4	−3.4	+.08	+.03	
1/2 wl	=	+ 28.75			+ 28.75		+.08		
V_{CD}	=	+ 27.95	−1.2	+4.1	+28.35	−3.4	+.08	+.03	−.03
	=	+.8	+1.2	− 4.1	+.4	+3.4	−.08	−.03	
1/2 wl	=	+ 28.75			+ 28.75				
V_{DC}	=	+ 29.55	+1.2	−4.1	+29.15	+3.4	−.08	−.03	+.03
$-M_{DE}-M_{ED}$	=	+ 295.1	+ 14.0	−43.0	+169.1	+ 157.5	+ 4.12	+ 4.620	
/l	=	+5.4	+.3	−.8	+ 3.1	+2.9	+.08	+.08	
1/2 wl	=	+ 27.1				+ 27.1			
V_{DE}	=	+32.5	+.3	−.8	+3.1	+30.0	+.08	+.08	−.08
	=	−5.4	−.3	+.8	− 3.1	− 2.9	−.08	−.08	
1/2 wl	=	+ 27.1				+ 27.1			
V_{ED}	=	+ 21.7	−.3	+.8	−3.1	+ 24.2	−.08	−.08	+.08

TABLE 14. REACTIONS

		Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8	
		W = 13000#	W = 4600#	W = 4600#	W = 4600#	W = 4600#	Q = 20000#	t = 80	t = 40	Maximum
V_{AB}	=	+ 343 000#	+ 133000#	− 13000#	+ 2000#	− 1000#	± 1000#	− 18000#	+ 9000#	+ 488000#
[illegible]	=	+ 497000	+ 165000	+ 13000	− 2000	+ 1000	± 1000	+ 18000	− 9000	+ 695000
V_{BC}	=	+ 427000	+ 21000	+ 141000	− 14000	+ 3000	± 1000	− 7000	+ 4000	+ 597000
V_{CB}	=	+ 369000	− 21000	+ 141000	+ 14000	− 3000	± 1000	+ 7000	− 4000	+ 532000
V_{CD}	=	+ 363000	− 6000	+ 19000	+ 130000	− 16000	± 2000	+ 7000	− 4000	+ 521000
V_{DC}	=	+ 385000	+ 6000	− 19000	+ 134000	+ 16000	± 2000	− 7000	+ 4000	+ 547000
V_{DE}	=	+ 423000	+ 1000	− 4000	+ 14000	+ 138000	± 2000	+ 18000	− 9000	+ 596000
V_{ED}	=	+ 281000	− 1000	+ 4000	− 14000	+ 112000	± 2000	− 18000	+ 9000	+ 408000

TABLE 15. STRESSES AT TOP OF COLUMNS

Column	AF	BG	CH	DJ	EK
Max. React.#	488000	1292000	1053000	1143000	408000
Col. Wt.	19000	19000	19000	19000	19000
Total Load#	507000#	1311000	1072000	1162000	427000
Col. Section	79" × $30\frac{1}{2}$"	79 × 32	79 × 32	79 × 32	79 × 30½
Col. Area	2360□"	2479	2479	2479	2360
Col. Steel	$18-1\frac{1}{4}$□	$26-1\frac{1}{4}$□	18-1□	$26-1\frac{1}{4}$□	$18-1\frac{1}{4}$□
Steel Area	28.1 □"	40.5	18.0	40.5	28.1
Direct Stress	190 #	430	390	380	160
$P/2$ %	.615	.84	.375	.84	.615
Max. B. Mom. '#	850000	1010000	273000	822000	624000
Max. Mom. "#	10200000	12120000	3280000	9860000	7490000
Eccentricity	20.1"	9.2	3.0	8.5	17.5
e/h	.66	.29	.094	.27	.57
M/bh^2f_c	.143	.146	.0685	.146	.142
Total Stress	1020 #	1080	620	880	760

TABLE 16. STRESSES AT BOTTOM OF COLUMNS.

Column	AF	BG	CH	DJ	EK
Max. Reaction	488000	1292000	1053000	1143000	408000
Col. Wt.	90000	95000	95000	95000	90000
Total Load#	578000	1387000	1148000	1238000	498000
Col. Section	79" × $30\frac{1}{2}$"	79 × 32	79 × 32	79 × 32	79 × $30\frac{1}{2}$
Col. Area□"	2360	2479	2479	2479	2360
Col. Steel	$18-1\frac{1}{4}$□	$26-1\frac{1}{4}$□	18-1□	$26-1\frac{1}{4}$□	$18-1\frac{1}{4}$□
Steel Area	28.1	40.5	18.0	40.5	28.1
Direct Stress	210#	456	420	406	181
$P/2$ %	.615	.84	.375	.84	.615
Max. Mom. '#	792000'#	1038000	253000	954000	618000
Max. Mom. "#	9500000"#	12460000	3040000	11450000	7420000
Eccentricity	16.5"	9.0	2.7	9.3	15.0
e/h	.54	.28	.085	.29	.49
M/bh^2f_c	.142	.146	.065	.146	.140
Total Stress	960 #	1100	610	1020	760

The method of Turneaure and Maurer in "Principles of Reinforced Concrete Construction" is used. (See pages 287 and 288, Second Edition, 1914.) Note that both maximum moments and reactions are used, although the combination of cases causing maximum moment will not cause maximum reaction. Since the bending moment due to temperature variations forms a large percentage of the total moments in the column and since the extreme temperature variations are of rare occurrence, the allowable stress in bending is raised from 750 pounds per square inch to 1000 or 1100 pounds per square inch.*

The stresses at the bottom of the column are shown in Table 16 . The stresses at the bottom of the curved haunch on the column are shown in Table 15 (called the top of the column). The moments used in this case are 25% less than the moments at the top of the columns, that is at the intersections with the center lines of the girders. The distance from these intersections to the bottom of the haunch is about 7½ feet, while the distance to the point of contraflexure is about 18 or 19 feet. Hence a reduction of 25% is conservative.

---

*Specifications for steel structures usually allow an increase of 30% in the unit stresses for loads not occuring often, such as wind loads.

H. MOMENT DIAGRAMS.

The moments at the joints from Table 12 were used to plot moment diagrams for the various cases of loading on the girder. The diagrams for the various cases were combined into diagrams showing maximum positive and negative moments. See drawing #4E Pre 143. From these diagrams the maximum moment at any point on a girder may be obtained.

I. DETAILS.

From the moment diagrams and the reactions the reinforcing steel may be easily calculated. The depths and widths of the girders are shown in the accompanying drawings. The details of the frame are shown in drawings #7210, 7211, 7212, 7214, 7230, 7231, 7252, 7253 and 7251.

J. REVIEW OF MOMENTS OF INERTIA.

The moments of inertia of the girders as now reinforced were calculated and found to be practically the same as the moments of inertia used in the beginning. Hence no changes in the details had to be made.

APPENDIX I.

BIBLIOGRAPHY.

The following is a list of books and articles dealing with the various methods of analyzing statically indeterminate structures. This list is far from being complete, as it contains only four references to books in foreign languages; **but it is fairly** representative of the books on this subject published in English.

M. Abe, "Analysis and Tests of Rigidly Connected Reinforced Concrete Frames", Bulletin #107, University of Illinois, Engineering Experiment Station, 1919.

E. S. Andrews, "Elastic Stresses in Structures", 1919. This book is a translation of the work of Castigliano mentioned below.

A. Bendixsen, "Die Methode der Alpha-Gleichungen zur Berechnung von Rahmenkonstruktionen", 1914.

Wm. Cain, "Determination of Stresses in Elastic Systems By Means of Least Work," Trans. Amer. Soc. Civ. Eng., 1891.

A. Castigliano, "Théorie de l'Equilibre des Systèmes Elastiques et ses Applications," 1879.

I. P. Church, "Mechanics of Internal Work."

F. V. Emperger, editor, "Handbuch Für Eisenbetonbau," 1st and 4th volumes of 1st edition.

O. Faber and P.G. Bowie, "Reinforced Concrete Design", 1912.

C. R. Grimm, "Secondary Stresses in Bridge Trusses", 1908.

A. H. Heller, "Stresses in Structures."

I. Hiroi, " Statically Indeterminate Structures"

G. A. Hool and N. C. Johnson, "Concrete Engineers' Handbook", 1918.

C. W. Hudson, "Statically Indeterminate Structures".

J. B. Johnson, C. W. Bryan and F. E. Turneaure "Modern Framed Structures", Vol. 2.

F. C. Kunz, "Secondary Stresses", Engineering News, Oct. 5th, 1911.

G. A. Maney, "Secondary Stresses and other Problems in Rigid Frames; a New Method of Solution", Bulletin #1, University of Minnesota, Studies in Engineering, 1915.

G. A. Maney, "A New Principle in Theory of Structures", Discussion, Proc. Am. Soc. Civ. Eng., August, 1919.

Menabrea, "Nouveau Principe Sur la Distribution des Tensions dans les Systèmes Elastiques", Comptes Rendu, 1858.

M. Merriman, "Mechanics of Materials.

D. A. Molitor, "Kinetic Theory of Engineering Structures."

C. M. Spofford, "Theory of Structures."

G. F. Swain, "Application of Principles of Virtual Velocities to Finding Deflection and Stresses of Frames", Journal Franklin Institute, 1883.

S. E. Thompson and E. Smulski, "Rigid Frames for Reinforced Concrete", "Cement Era, March 1913.

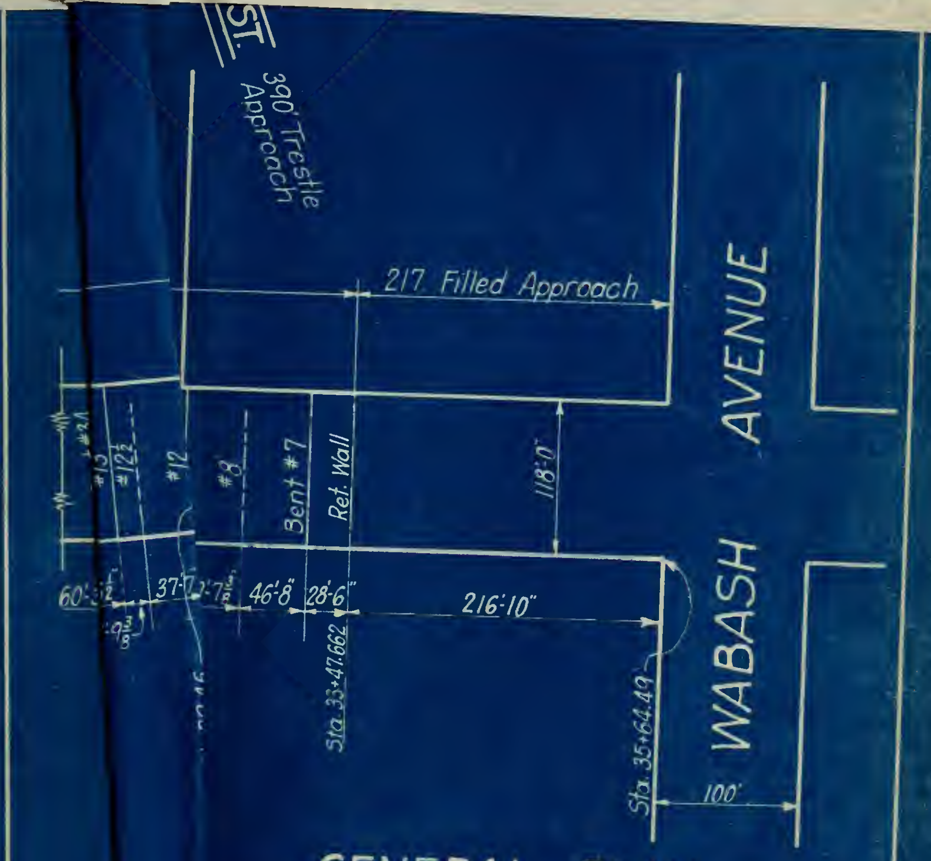
W. M. Wilson and G. A. Maney, "Wind Stresses in the Steel Frames of Office Buildings", Bulletin #80, University of Illinois, Eng. Exp. Sta., 1915.

W. M. Wilson, F. E. Richart and C. Weiss, "Analysis of Statically Indeterminate Structures by the Slope Deflection Method, " Bulletin #108, Univeristy of Illinois, Engineering Experimental Station, 1919.

APPENDIX II.

Note to Chapter II.

After Chapter II was printed the writer found in C. R. Grimm's "Secondary Stresses"(1908) a proof of the fundamental equation of "Mohr's Method" for secondary stresses. The equation is derived by a graphical-analytical method making use of the equilibrium polygon; also of EI and M as forces.

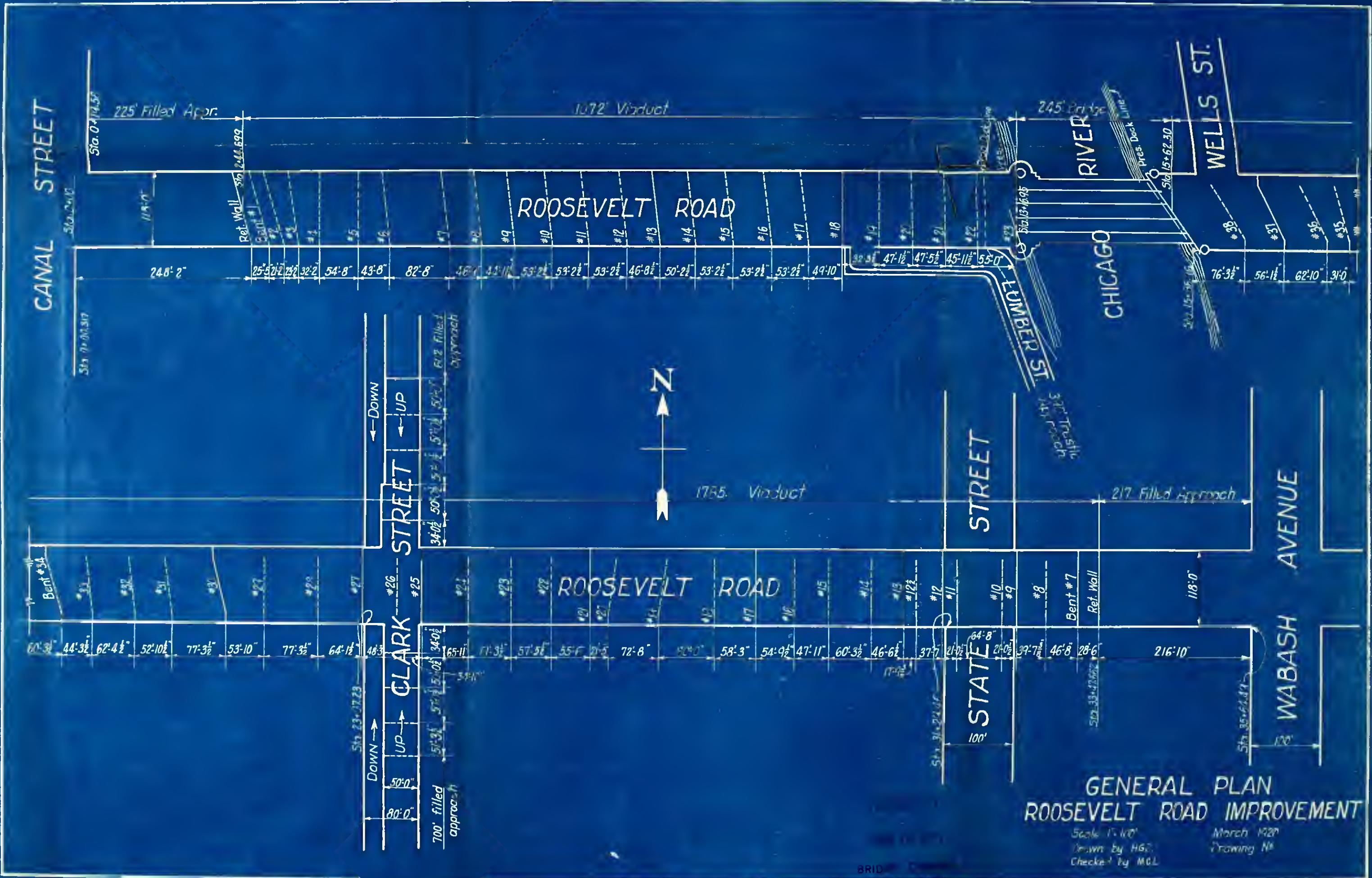


GENERAL PLAN
ROOSEVELT ROAD IMPROVEMENT

Scale 1" = 100'
Drawn by H.G.B.
Checked by M.C.L.

March 1920
Drawing No.

4E per 429

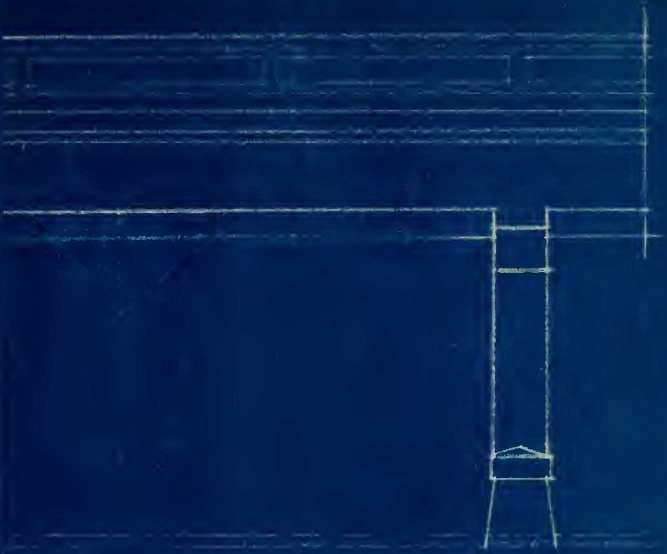


GENERAL PLAN
ROOSEVELT ROAD IMPROVEMENT

Scale 1"=100'
Drawn by HGZ
Checked by MCL

March 1920
Drawing No

4E per 429



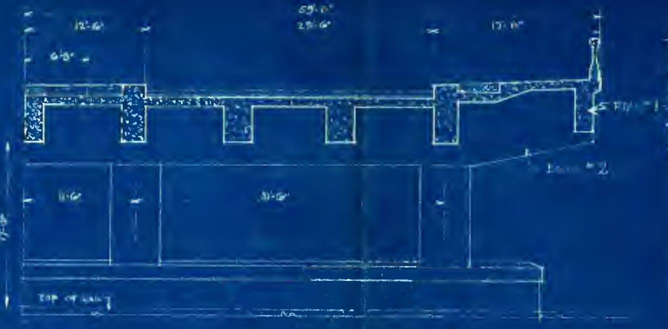
ICAL ELEVATION (OLD DESIGN).

Scale 1/8"=1'-0"

12 Street Viaduct
Concrete Viaduct
Exhibit "A"
1/8"=1'-0"

#E Pc 409
E-6

23 7/20-19



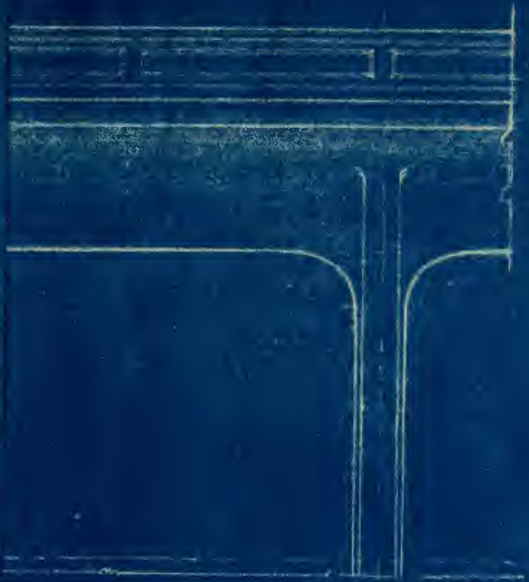
TYPICAL SECTION (OLD DESIGN) EXHIBIT A
SCALE 1/8"=1'-0"

TYPICAL ELEVATION (OLD DESIGN)
SCALE 1/8"=1'-0"

(See Exhibit A)

12th Street Viaduct
Concrete Viaduct
Exhibit "A"
1/8"=1'-0"
[illegible] 3/22/11

4 E Pl. 409
E-6



LEVATION (NEW DESIGN).

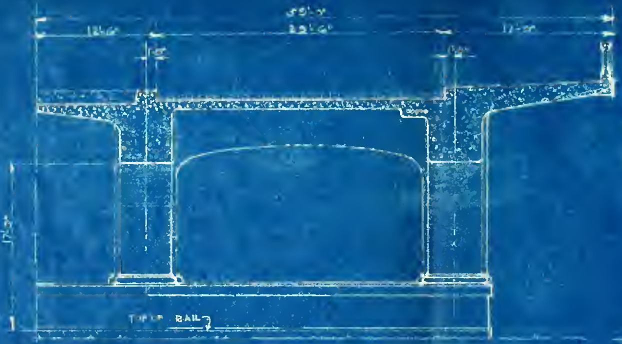
SCALE $\frac{1}{8}$" = 1'-0"

[illegible] 18th Street. Viaduct

TITLE Concrete Viaduct
Exhibit 8. New Type
1/8" = 1'-0"

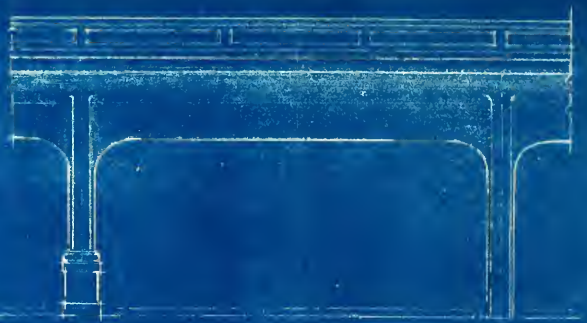
25 3/26-19

AE Pr 403
E-6



TYPICAL SECTION (NEW DESIGN) EXHIBIT B.

SCALE 1/8" = 1'-0"



TYPICAL ELEVATION (NEW DESIGN).

SCALE 1/8" = 1'-0"

JOB 12th Street Viaduct

TITLE Concrete Viaduct
Exhibit B. New Design
1/8" = 1'-0"

3/22/14

4ER 403
E-6

T 29122
q624 G89

Grodsky, Morris

Design of Roosevelt road DEMCO-216
viaduct

T 29122 G89
q624

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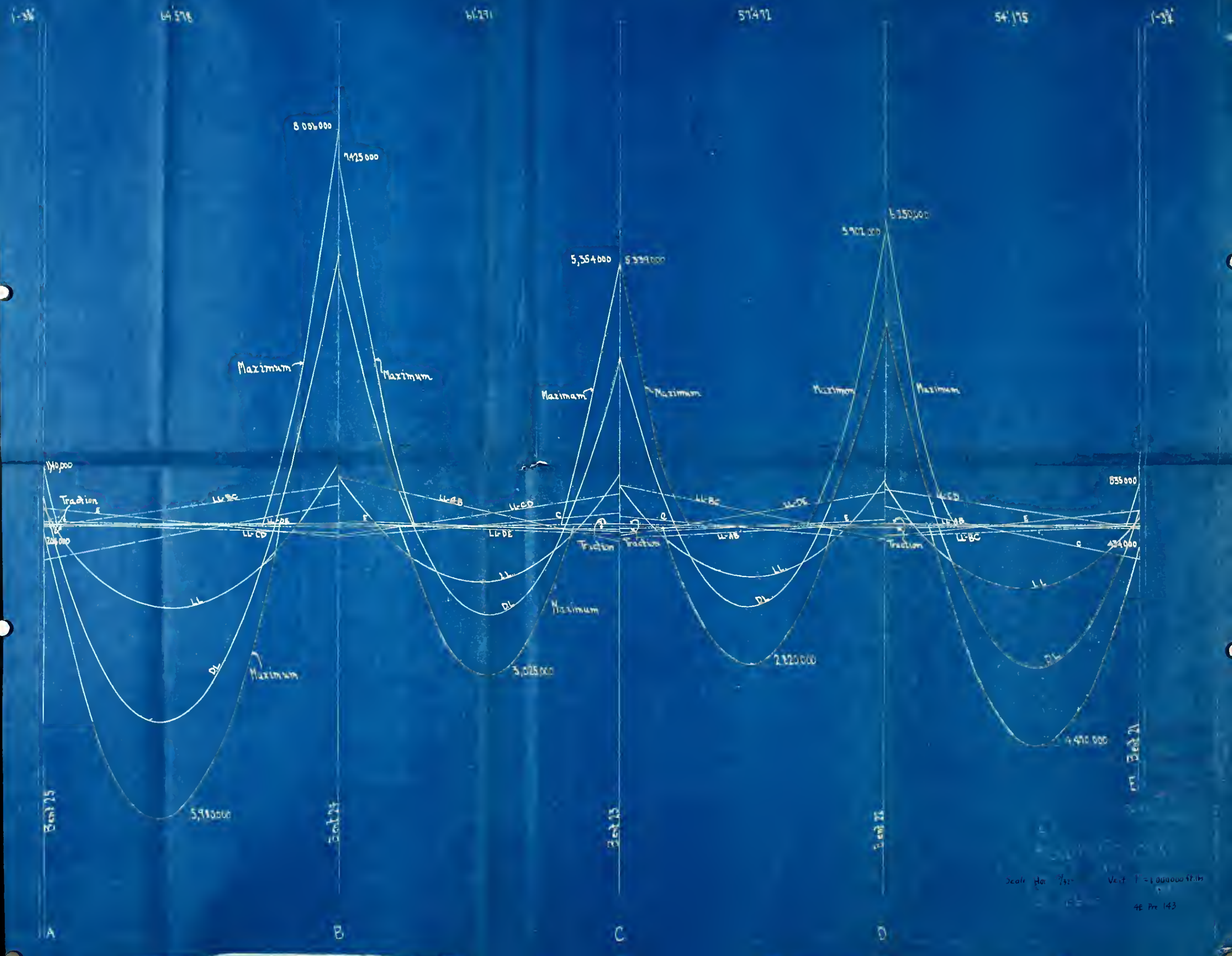
1-3¾
64.578
61.291
57.472
54.175
1-3¾
8 006 000
7,425 000
5,354 000
5,339,000
5,902,000
6,250,000
Maximum
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Traction
LL-BC
LL-AB
LL-CD
LL-DE
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LL-DE
LL-BC
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LL-AB
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3,025,000
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Scale Hor 1/32"
Vert 1" = 1,000000 ft.lbs
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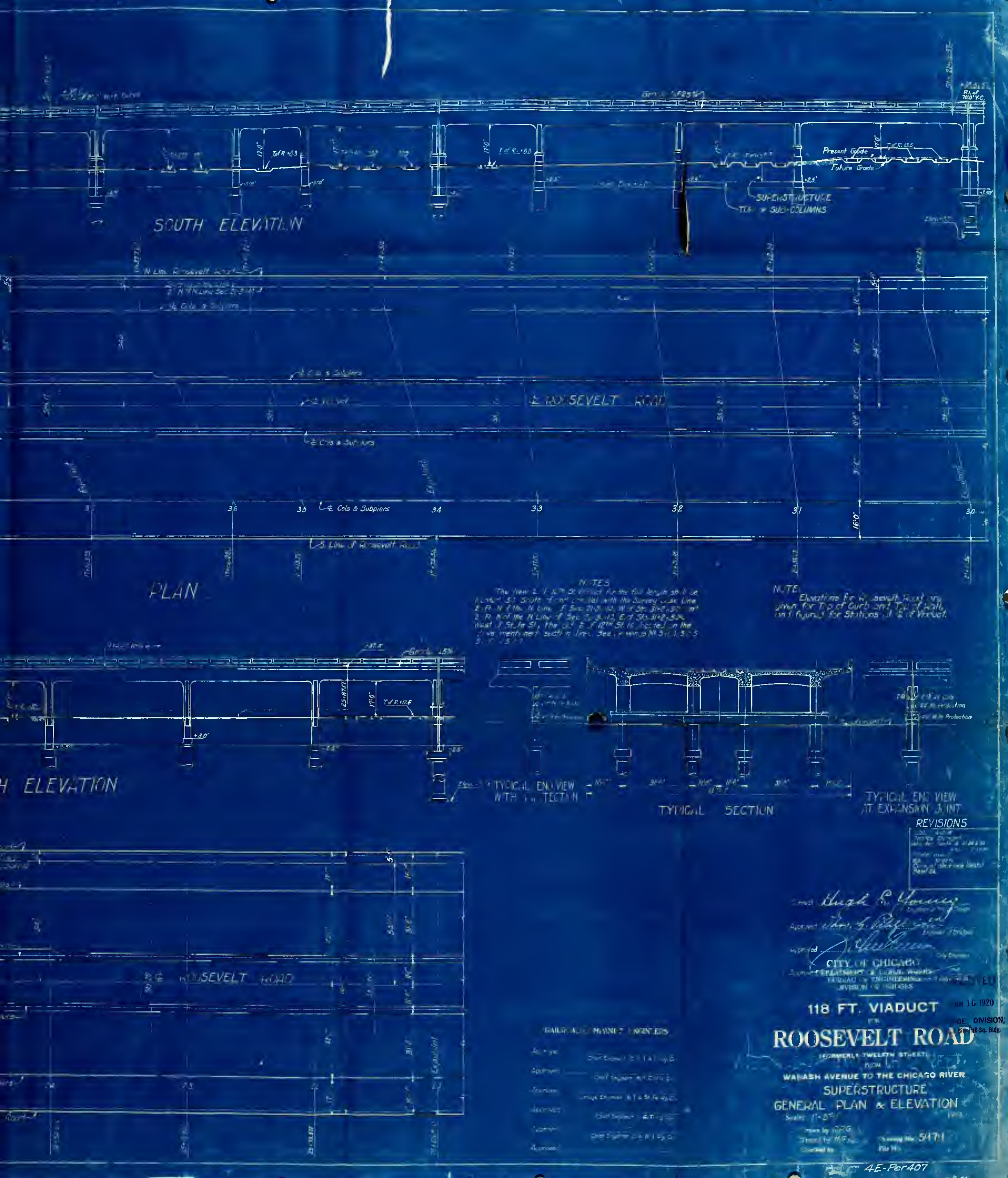


NOTES

The new ℄ of 12th St. Viaduct for the full length shall be located 3.2' South of and parallel with the Survey base line 2 ft. N of the N. Line of Sec. 21-39-14, W of Sta. 31+2.50 and 2 ft. N of the N. Line of Sec. 22-39-14, E of Sta. 31+2.50. West of State St., the old ℄ of 12th St. is located on the above mentioned section lines. See drawings No. [illegible]

NOTE

Elevations for Roosevelt Road are given for Top of Curb and Top of Walk, and figures for Stations on ℄ of Viaduct.

Correct Hugh E. Young
Engineer of ~~Plans~~

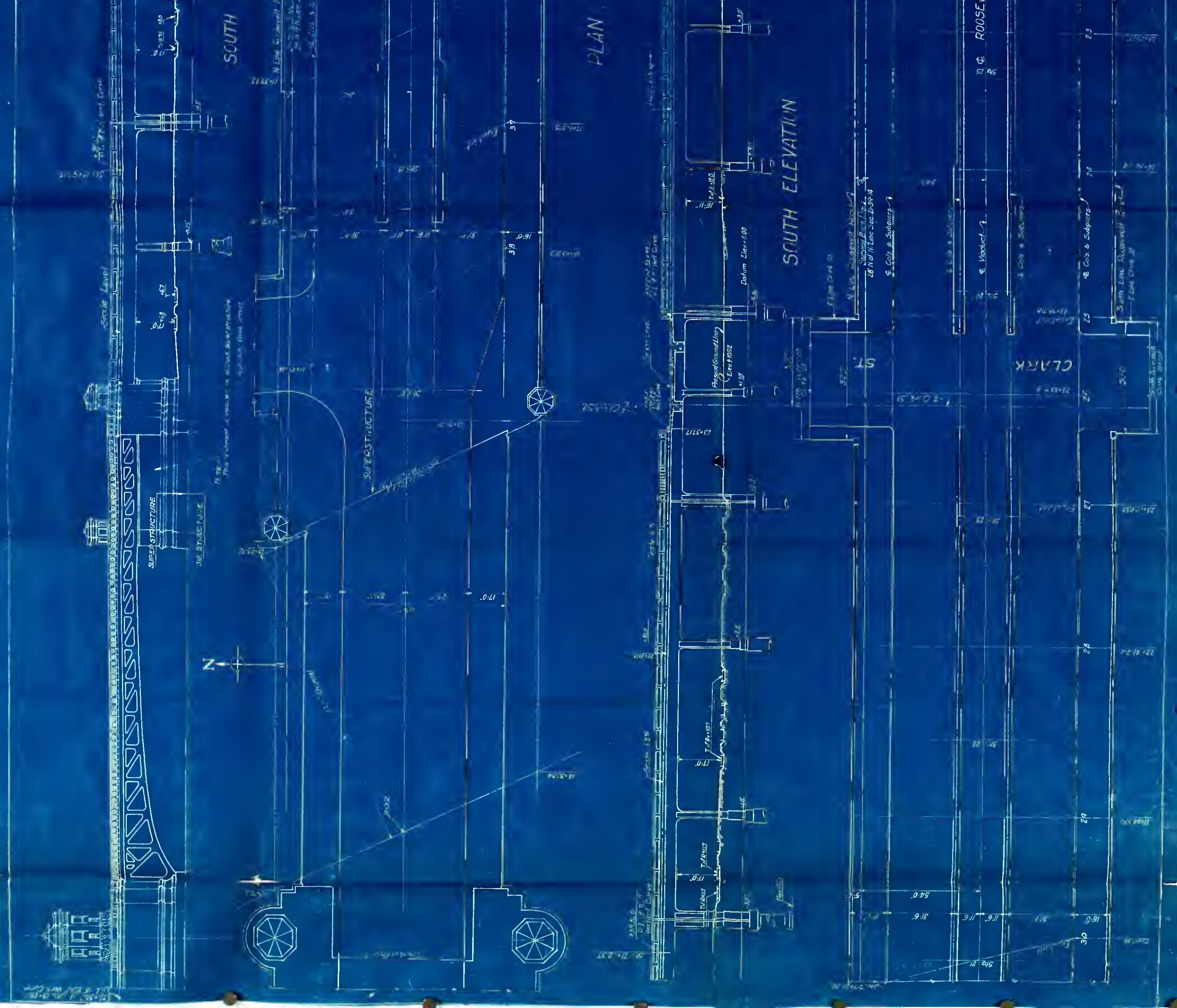
Approved
Engineer of Bridges

Approved
City Engineer

118 FT. VIADUCT
FOR
ROOSEVELT ROAD
(FORMERLY TWELFTH STREET)
FROM
WABASH AVENUE TO THE CHICAGO RIVER
SUPERSTRUCTURE
GENERAL PLAN & ELEVATION
Scale: 1" = 20'

Drawn by G.P.G.
Traced by M.F.
Checked by
Drawing No. 5471
File No.

16 1920


SOUTH
PLAN
SOUTH ELEVATION
N. Line Roosevelt Rd.
Grade Level
SUPERSTRUCTURE
SUBSTRUCTURE
SUPERSTRUCTURE
N
Datum Elev + 0.00
Present Ground Line Elev + 10.52
Survey Base Line 28' N of N. Line Sec. 21-39-14
℄ Cols & Subpiers
℄ Viaduct
℄ Cols & Subpiers
South Line Roosevelt Road
℄ ROOSE
CLARK
ST.
23
24
25
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29
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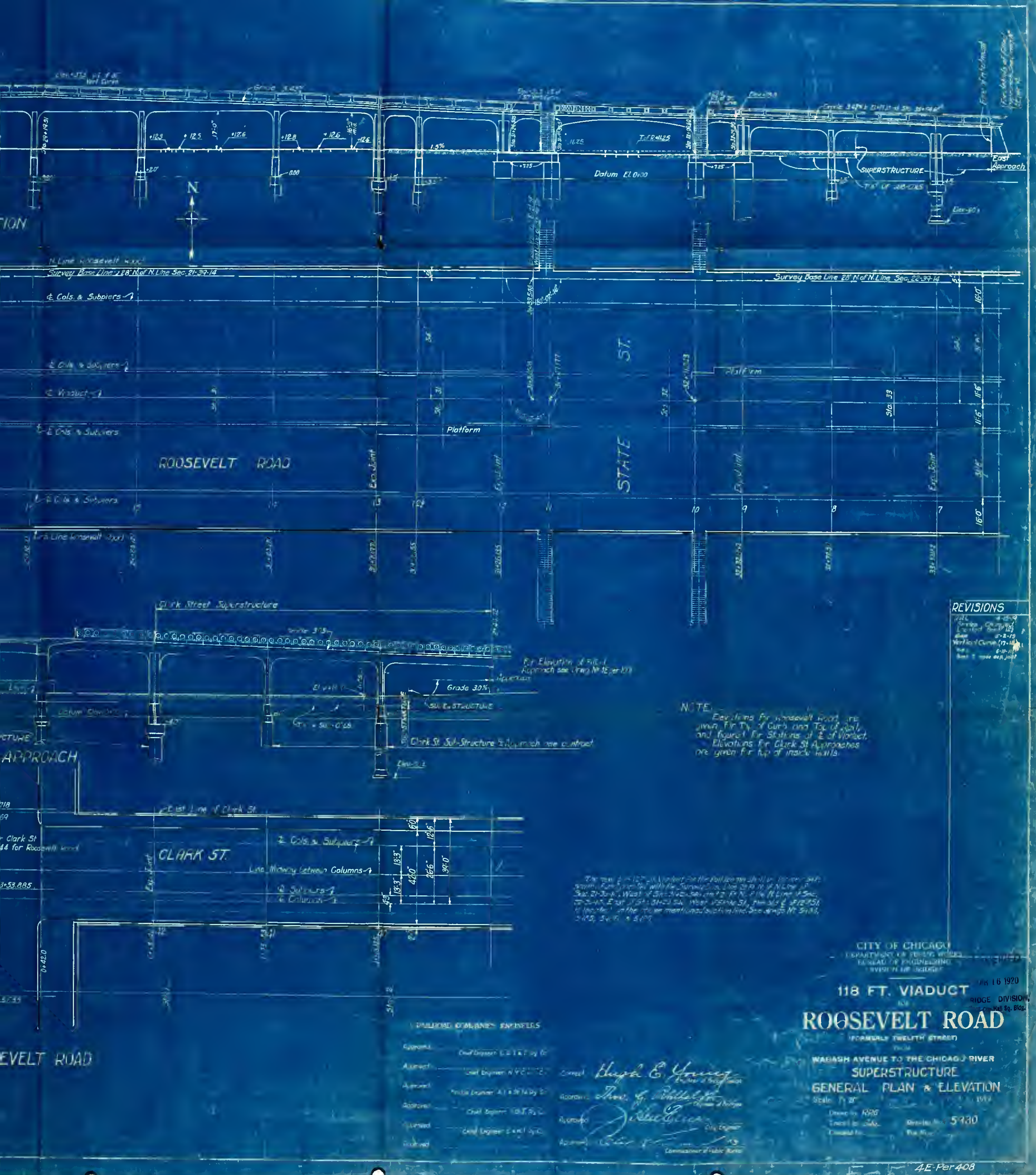
17'-0"
16'-0"
54'-0"
31'-6"
11'-6"
11'-6"



NOTE.
Elevations for Roosevelt Road are given for Top of Curb and Top of Rail, and figured for Stations at ℄ of Viaduct.
Elevations for Clark St. Approaches are given for top of inside walls.

CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING
DIVISION OF BRIDGES

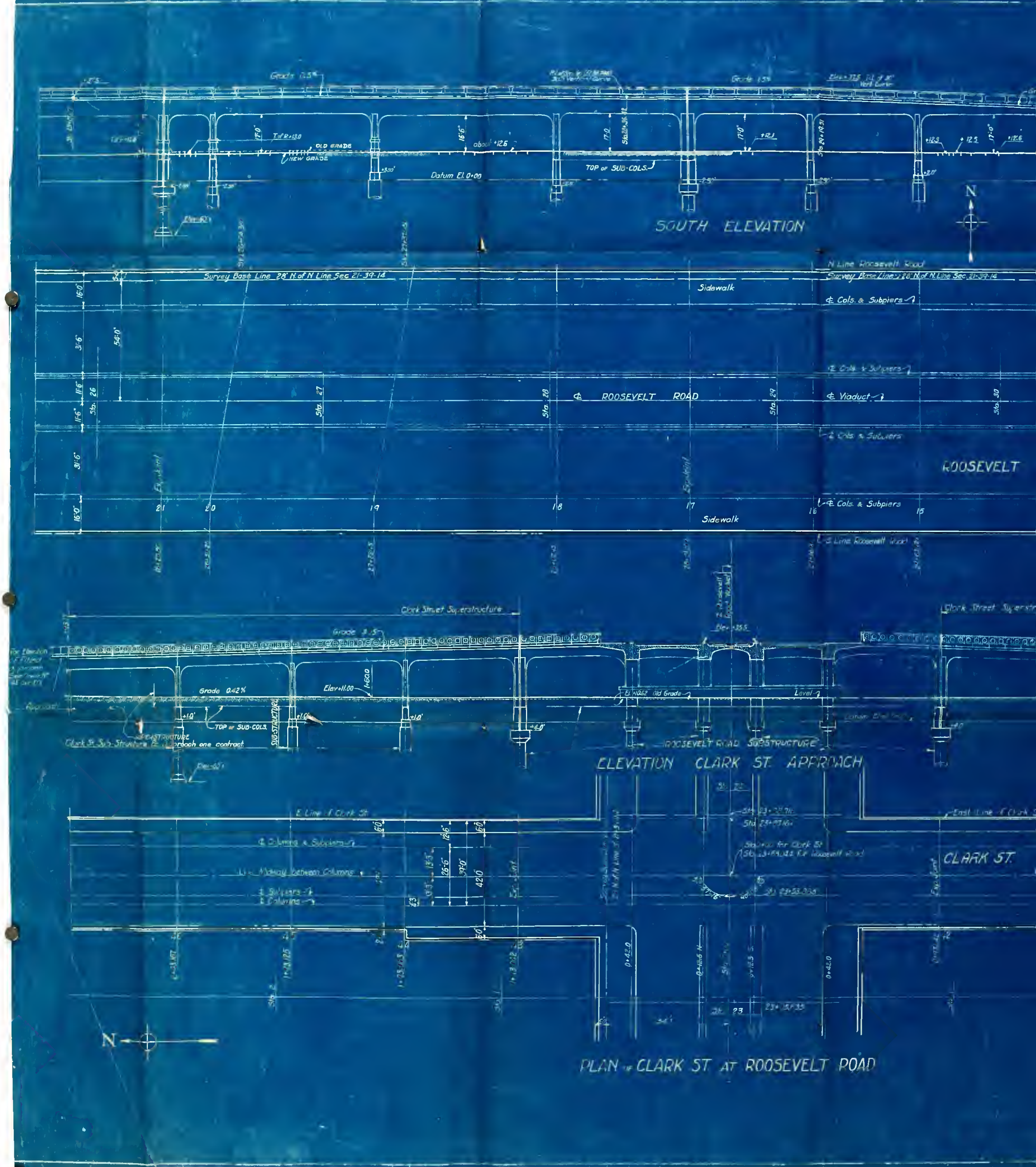
118 FT. VIADUCT
IN
ROOSEVELT ROAD
(FORMERLY TWELFTH STREET)
FROM
WABASH AVENUE TO THE CHICAGO RIVER
SUPERSTRUCTURE
GENERAL PLAN & ELEVATION

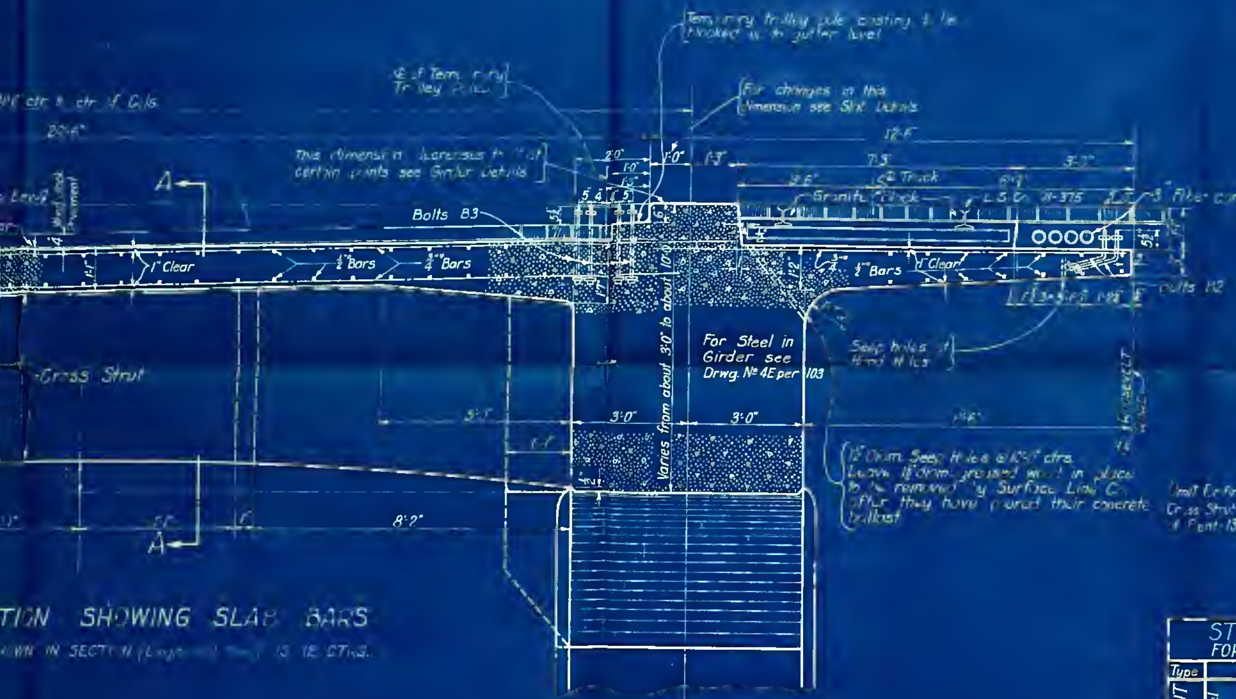
Scale 1" = 20'
Drawn by RRG
Drawing No. 5930

RAILROAD COMPANIES' ENGINEERS
Approved Chief Engineer
Approved Chief Engineer N.Y.C.
Approved Chief Engineer
Approved Chief Engineer
Approved Chief Engineer
Approved

Correct Hugh E. Young
Approved Thos. G. Pihlfeldt
Engineer of Bridges
Approved
City Engineer
Approved
Commissioner of Public Works

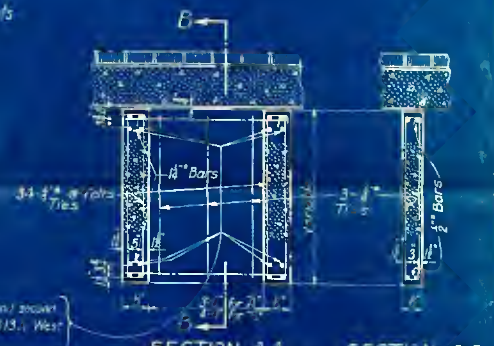
16 1920
BRIDGE DIVISION
City Hall Sq. Bldg.


SOUTH ELEVATION
Grade 0.5%
Grade 1.5%
OLD GRADE
NEW GRADE
about +12.6
Datum El. 0+00
TOP of SUB-COLS.
+12.1
+12.5
+12.6
17'-0"
16'-6"
N
N Line Roosevelt Road
Survey Base Line 28' N. of N Line Sec. 21-39-14
Sidewalk
℄ Cols. & Subpiers
℄ Viaduct
℄ ROOSEVELT ROAD
ROOSEVELT
Sta. 26
Sta. 27
Sta. 28
Sta. 29
Sta. 30
16'-0"
31'-6"
11'-6"
54'-0"
21
20
19
18
17
16
15
S Line Roosevelt Road
Clark Street Superstructure
Grade 3.5%
Grade 0.42%
Elev +11.00
Elev +35.5
El. 10.62 Old Grade
Level
+1.0'
+4.0'
TOP of SUB-COLS.
SUB-STRUCTURE
Clark St. Sub-Structure & Approach one contract
ROOSEVELT ROAD SUBSTRUCTURE
ELEVATION CLARK ST. APPROACH
E Line of Clark St.
East Line of Clark St.
℄ Columns & Subpiers
Midway between Columns
℄ Subpiers
℄ Columns
6'-0"
12'-6"
13'-3"
26'-6"
39'-0"
42'-0"
Sta. 2
Sta. 23
0+42.0
CLARK ST.
PLAN of CLARK ST. AT ROOSEVELT ROAD
N


ctr. to ctr. of Cols
20'4"
Temporary trolley pole casting to be hooked up to gutter level
℄ of Temporary Trolley Poles
For changes in this dimension see Slab Details
This dimension increases to 1'0" at certain points see Girder Details
12'F
7'5"
3'2"
2'0"
1'0"
1'3"
2'6"
℄ Truck
6'0"
Granite Block
L.S.G. 11-375
3" Fibre conduits
Bolts B3
A
1" Clear
½" Bars
¾" Bars
¾" Bars
½" Clear
Bolts B2
Cross Strut
For Steel in Girder see Drwg. Nº 4E per 103
Varies from about 3'0" to about 10'0"
Seep holes at Pier Walls
5'7"
3'0"
3'0"
14'6"
1'1"
8'2"
A
1½" Diam. Seep Holes 6'10½" ctrs. Leave 1½" diam. greased wood in place to be removed by Surface Line Co. after they have placed their concrete ballast

TION SHOWING SLAB BARS

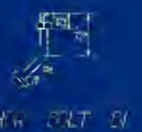
WN IN SECTION (Longitudinal Bars) IS 12 CTRS.


B
B
1¼" Bars
34-⅜" Ø Ties
3-⅞" Ties
¾" Bars
Limit for first and second Cross Strut CS13A West of Pent 13.

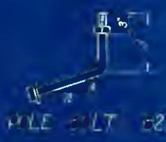
SECTION A-A SECTION B-B
DETAIL OF CROSS STRUT

STRAIGHT STEEL FOR CROSS STRUTS					
Type		Nº Reqd	Size	Length	Remarks
CROSS STRUT "A"	Steel for One	16	1¼"	32'-6"	Except for 1st & 2nd Strut CS13A W of Pent 13 See note above
		8	½"	4'-0"	
	Total Steel for		1¼"	32'-6"	
			½"	4'-0"	
CROSS STRUT "B"	Steel for One	16	1¼"	32'-6"	
		8	½"	5'-0"	
	Total Steel for			32'-6"	
				5'-0"	

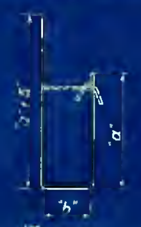
LIST OF TIES FOR CROSS STRUTS						
CROSS STRUTS		TIES				
Nº	Mark	Nº	Size	"a"	"b"	Length
2	CS6A	136 12	¼" ·	3'6" ·	5" 3"	7'-10" 7'-8"
2	CS7A	136 12	· ·	· ·	5" 3"	7'-10" 7'-8"
2	CS8A	136 12	· ·	· ·	5" 3"	7'-10" 7'-8"
2	CS12A	136 12	· ·	· ·	5" 3"	7'-10" 7'-8"
6	CS13A	408 36	· ·	1'-9" ·	5" 3"	4'-4" 4'-0"
6	CS14A	408 32	· ·	3'-0" ·	5" 3"	[illegible]
2	CS15A	136 12	· ·	3'-1" ·	5" 3"	7'-0" 6'-8"
2	CS15B	136 12	· ·	3'-6" ·	5" 3"	8'-2" 7'-10"
4	CS16A	272 24	· ·	4'-6" ·	5" 3"	9'-10" 9'-6"
2	CS17A	136 12	· ·	5'-1" ·	5" 3"	11'-0" 10'-8"
2	CS17B	136 12	· ·	5'-1" ·	5" 3"	11'-0" 10'-8"
6	CS18A	408 36	· ·	· ·	5" 3"	11'-0" 10'-6"
6	CS19A	408 36	· ·	5'-3" ·	5" 3"	11'-4" 11'-0"
4	CS21A	272 24	· ·	5'-2" ·	5" 3"	11'-2" 10'-10"
2	CS22A	136 12	· ·	5'-1½" ·	5" 3"	11'-0" 10'-8"
2	CS22B	136 12	· ·	5'-0" ·	5" 3"	10'-10" 10'-6"
2	CS23A	136 12	· ·	4'-9" ·	5" 3"	10'-4" 10'-0"
2	CS23B	136 12	 ·	4'-9" ·	5" 3"	10'-[illegible] 9'-10"
4	CS24A	272 24	· ·	4'-6" ·	5" 3"	9'-10" 9'-6"



ANCHOR BOLT B1



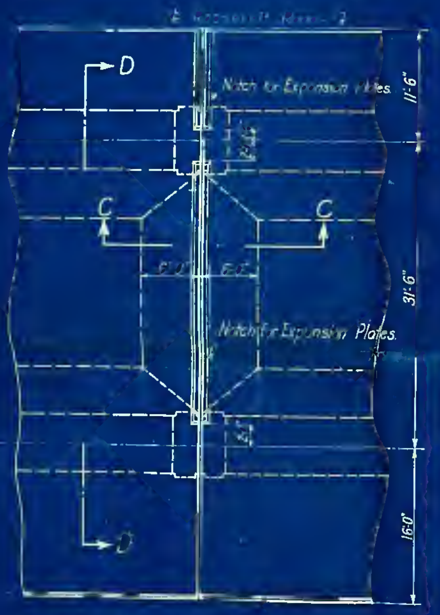
POLE BOLT B2



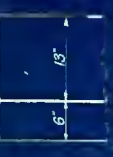
⅜" TIES
Bend hooks to suit, both

MIX: 1:2:4

BILL OF BOLTS				
Nº Reqd	Description	Mat'l	Mark	Remarks
1375	Bent Bolts 1" dia. x 11" long	Steel	B1	With Sq. Hd., Hex. Nut & Cut Washer
168	Bent Bolts 1¼" dia. x 1'-6" long	Steel	B2	"
168	Bolts 1¼" dia. x 1'-6" long	Steel	B3	"



PLAN AT EXPANSION JOINT
NORMAL TO ℄ OF VIADUCT



Hugh E. Young

CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING
DIVISION OF BRIDGES

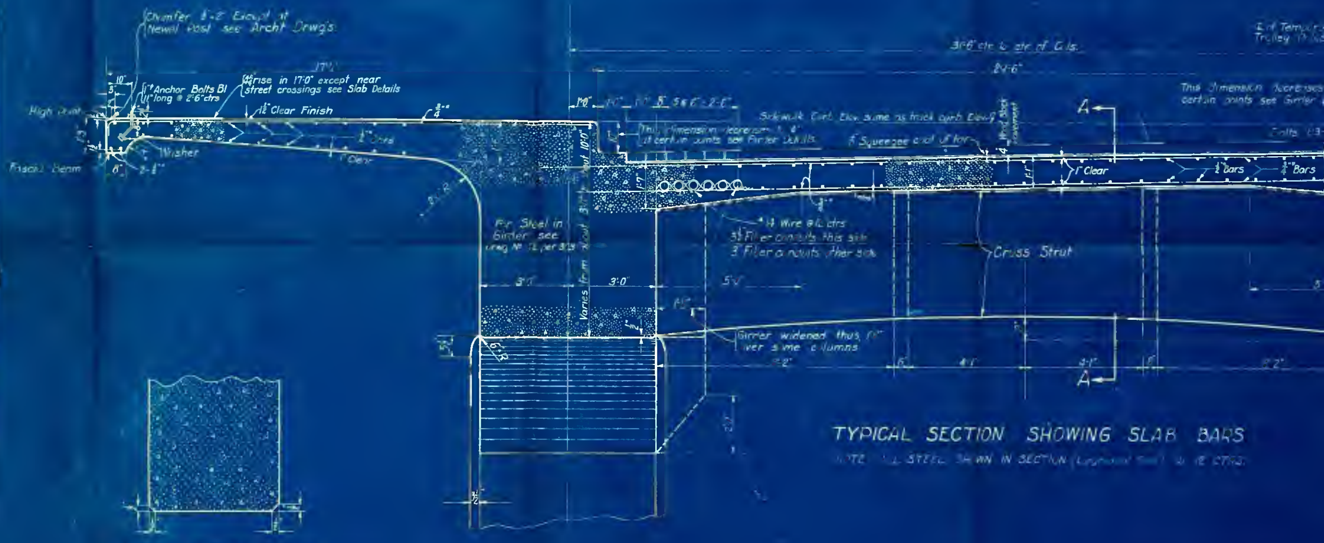
118 FT. VIADUCT

ROOSEVELT ROAD
(FORMERLY TWELFTH STREET)

WABASH AVENUE TO THE CHICAGO RIVER
SUPERSTRUCTURE
SLAB & CROSS STRUT DETAIL

1 1920
DGE VISION

KUK & GAS
H.E.L.
7210
H.E.B.

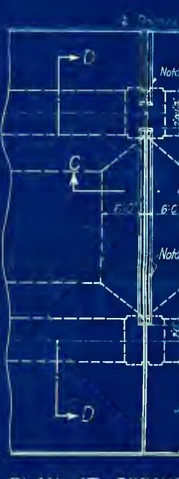

Chamfer 1"×2" Except at Newel Post see Arch't Drwg's.
17'0"
10"
5"
1" Anchor Bolts B1 1'1" long @ 2'6" ctrs
1/4" rise in 17'0" except near street crossings see Slab Details
1/2" Clear Finish
3/4"
High Point
Fascia Beam
Washer
6"
2-1/2"
1/2" ctrs
1" Clear
8'1" R
1'0"
Sidewalk Curb Elev. same as track curb Elev.
This dimension decreases to 8" at certain points see Girder Details
Squeegee coat of tar
Wood block pavement
10'0"
1'7"
Varies from about 3'1" to about 10'0"
For Steel in Girder see Dwg No. 12 per 819
3'0"
3'0"
5'V"
#14 Wire @ 6" ctrs
3 1/2" Fiber conduits this side
3" Fiber conduits other side
Girder widened thus 1'0" over some columns
1'2"
1/2"
6" R
3'6" ctr. to ctr. of Cols.
24'6"
A
A
1" Clear
1/2" Bars
3/4" Bars
Cross Strut
6"
4'1"
4'1"
6"
8'2"
4"
1'1"
Ⓒ of Temporary Trolley Track
This dimension decreases at certain points see Girder

TYPICAL SECTION SHOWING SLAB BARS

NOTE ALL STEEL SHOWN IN SECTION (Longitudinal Steel) @ 12 CTRS.

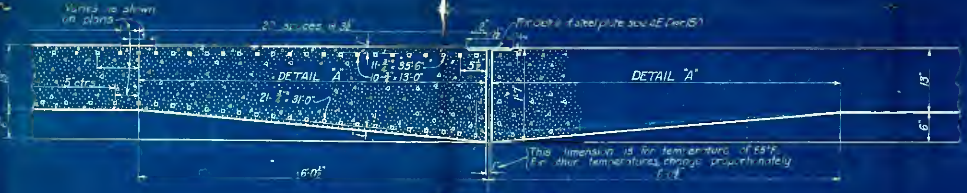
SKETCH SHOWING CHAMFER

All corners to be chamfered as shown in this sketch, except where otherwise noted



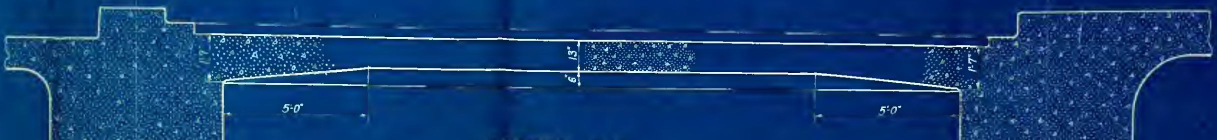
PLAN AT EXPANS

NORMAL TO Ⓒ OF

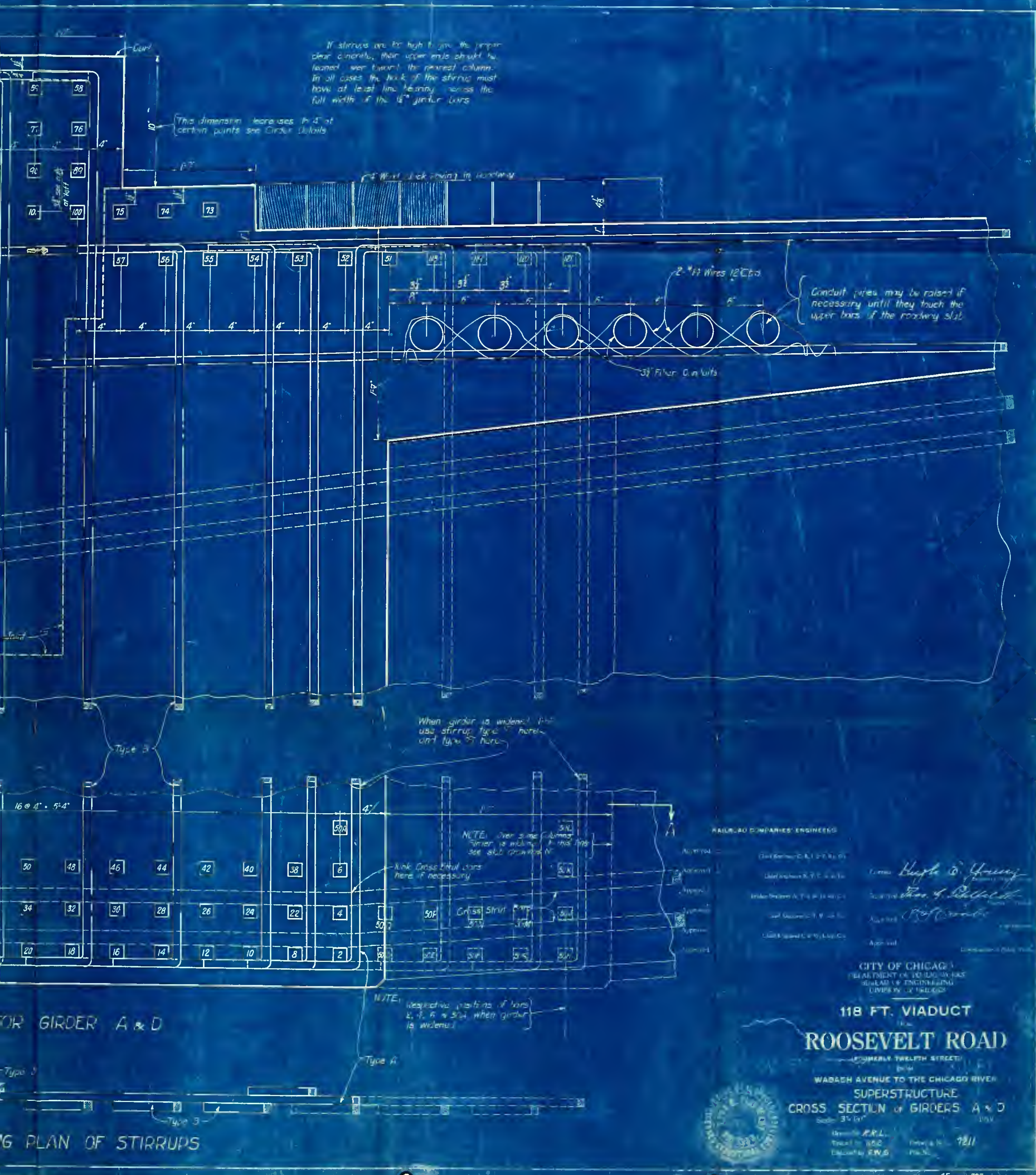


SECTION C-C

Showing increased slab thickness at Expansion Joint occurring only between Columns A & B and C & D



SECTION D-D


If stirrups are too high to give the proper clear concrete, their upper ends should be leaned over toward the nearest column. In all cases the hook of the stirrup must have at least line bearing across the full width of the 1¼" girder bars
This dimension decreases to 4" at certain points see Girder Details
4" Wood block paving in Roadway
Curb
10"
4"
4½"
57
56
55
54
53
52
51
58
59
76
89
90
100
73
74
75
2-#14 Wires 12" Ctrs
Conduit pipes may be raised if necessary until they touch the upper bars of the roadway slab
3½" Fiber Conduits
3½"
8"
6"
1'-4"
Type 3
When girder is widened use stirrup type here and type here
16 @ 4" = 5'-4"
NOTE. Over some columns Girder is widened to this line see slab drawing
Kink Cross Strut bars here if necessary
Cross Strut
50A
50F
50
48
46
44
42
40
38
6
34
32
30
28
26
24
22
4
20
18
16
14
12
10
8
2
NOTE. Respective positions of bars 2, 4, 6 & 50A when girder is widened
Type A
A
OR GIRDER A & D
G PLAN OF STIRRUPS

RAILROAD COMPANIES' ENGINEERS

Approved
Approved
Approved
Approved
Approved
Approved

Hugh E. Young
Thos. G. Pihlfeldt

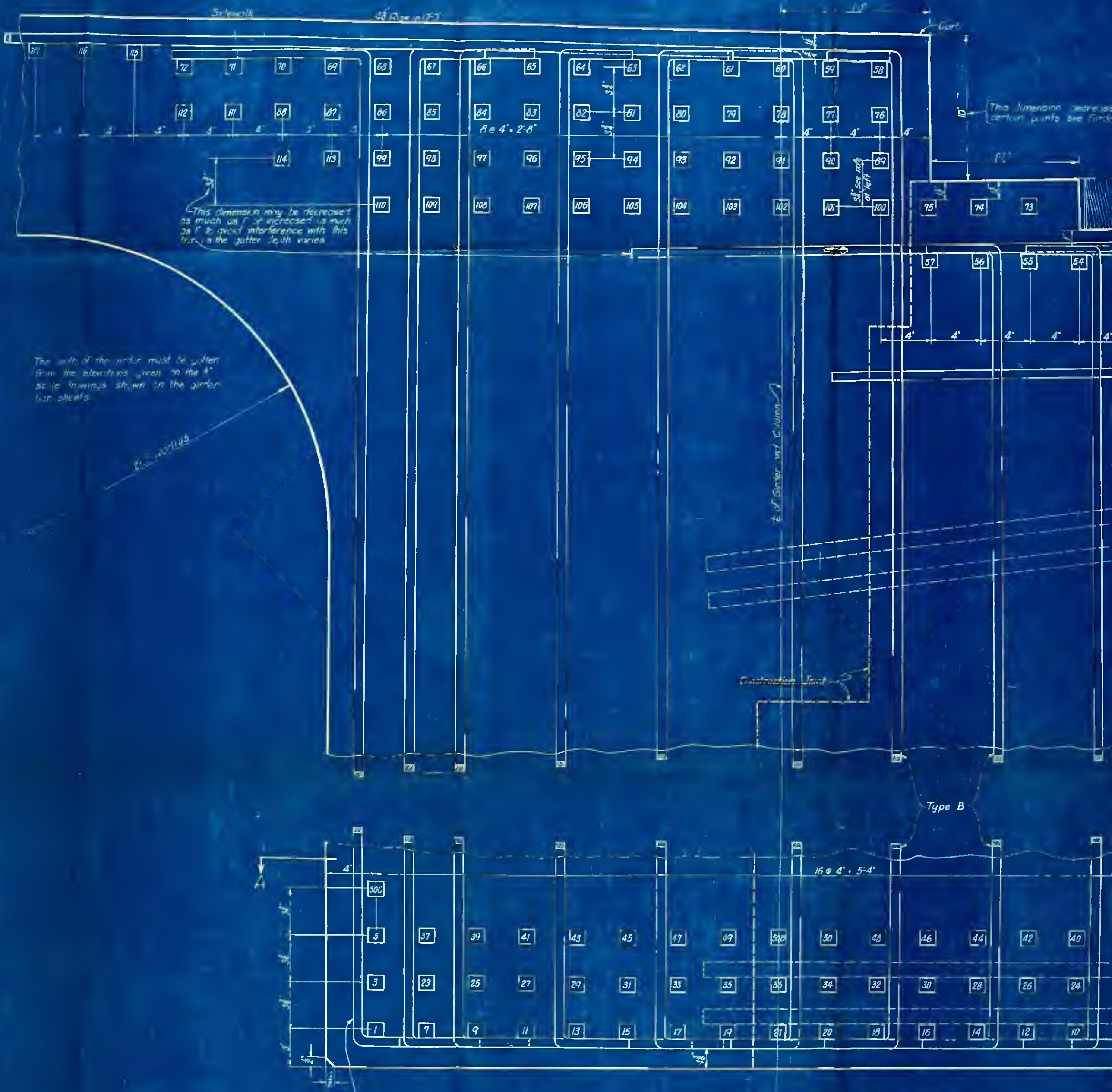
Approved

CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING
DIVISION OF BRIDGES

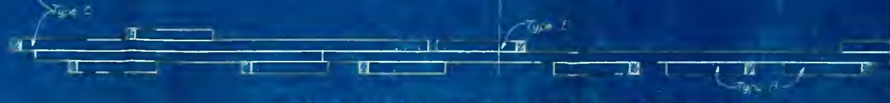
118 FT. VIADUCT
ROOSEVELT ROAD
(FORMERLY TWELFTH STREET)
WABASH AVENUE TO THE CHICAGO RIVER
SUPERSTRUCTURE
CROSS SECTION of GIRDERS A & D
Scale 3" = 1'-0"

Drawn by R.R.L.
Traced by W.E.B.
Checked by E.W.S.
Drawing No. 7811

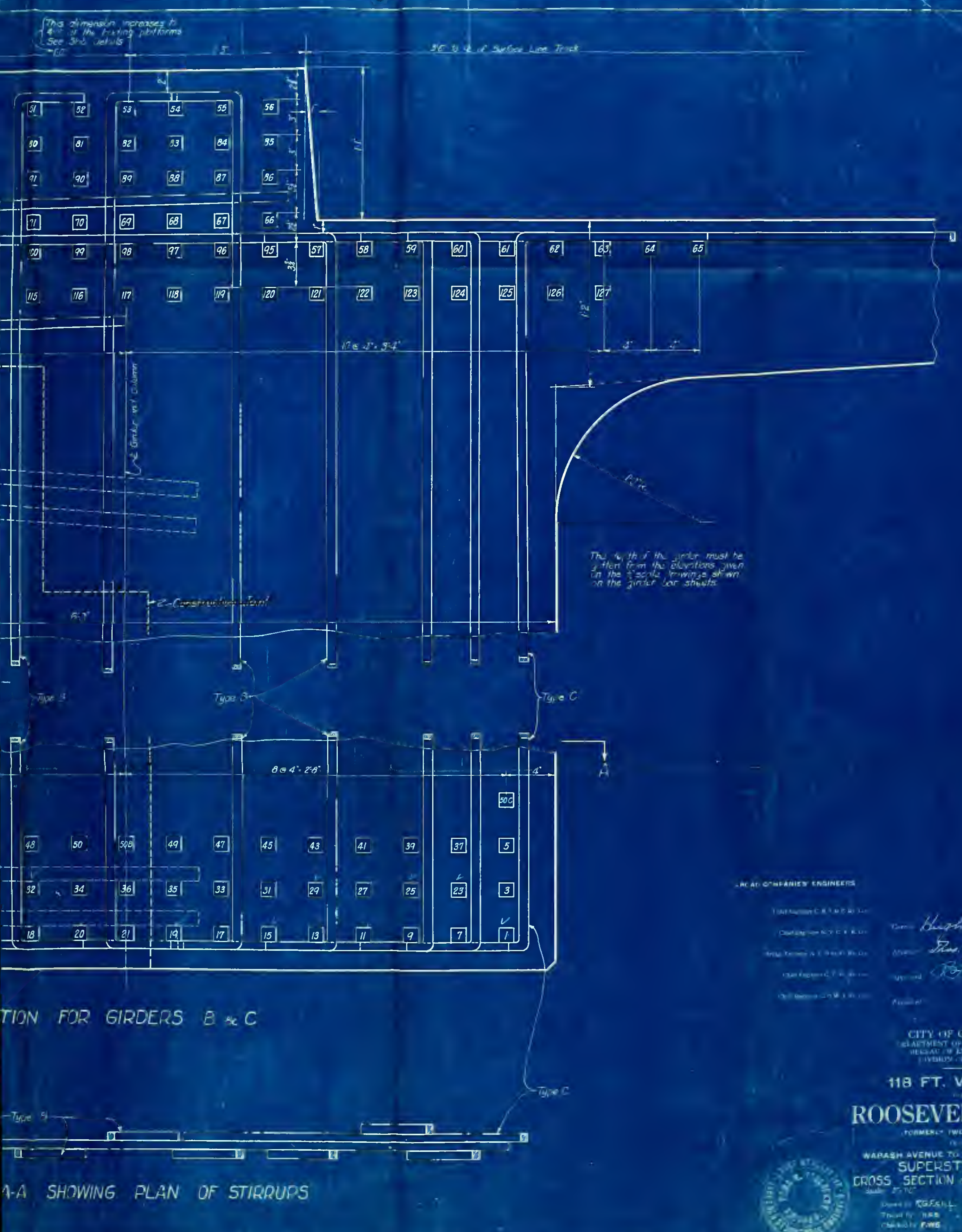
4E per-303


Sidewalk
Rise in
Curb
This dimension decrease
certain points see
10"
8 @ 4" = 2'-8"
see note at left
This dimension may be decreased as much as 1" or increased as much as 1" to avoid interference with this bar as the gutter depth varies
The width of the girder must be gotten from the elevations given on the ¼" scale drawings shown on the girder bar sheets.
radius
℄ of Girder and Column
Construction Joint
Type B
16 @ 4" = 5'-4"
A

KEY SECTIONS FOR GIRDER A & D



SECTION A-A SHOWING PLAN OF STIRRUPS


This dimension increases to
4'-6" at the loading platforms
See 3rd details
3'-6" to ℄ of Surface Line Track
11'
10 @ 4'-3'-4"
℄ Girder on 1 Column
℄-Construction Joint
Type B
Type B
Type C
Type C
Type B
8 @ 4'-2'-8"
4'
A
50C
50B
The depth of the girder must be gotten from the elevations given on the ¼" scale drawings shown on the girder bar sheets.

TION FOR GIRDERS B & C

A-A SHOWING PLAN OF STIRRUPS

RAILROAD COMPANIES' ENGINEERS

Hugh E. Young
Thos. G. Pihlfeldt

CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING
DIVISION OF BRIDGES

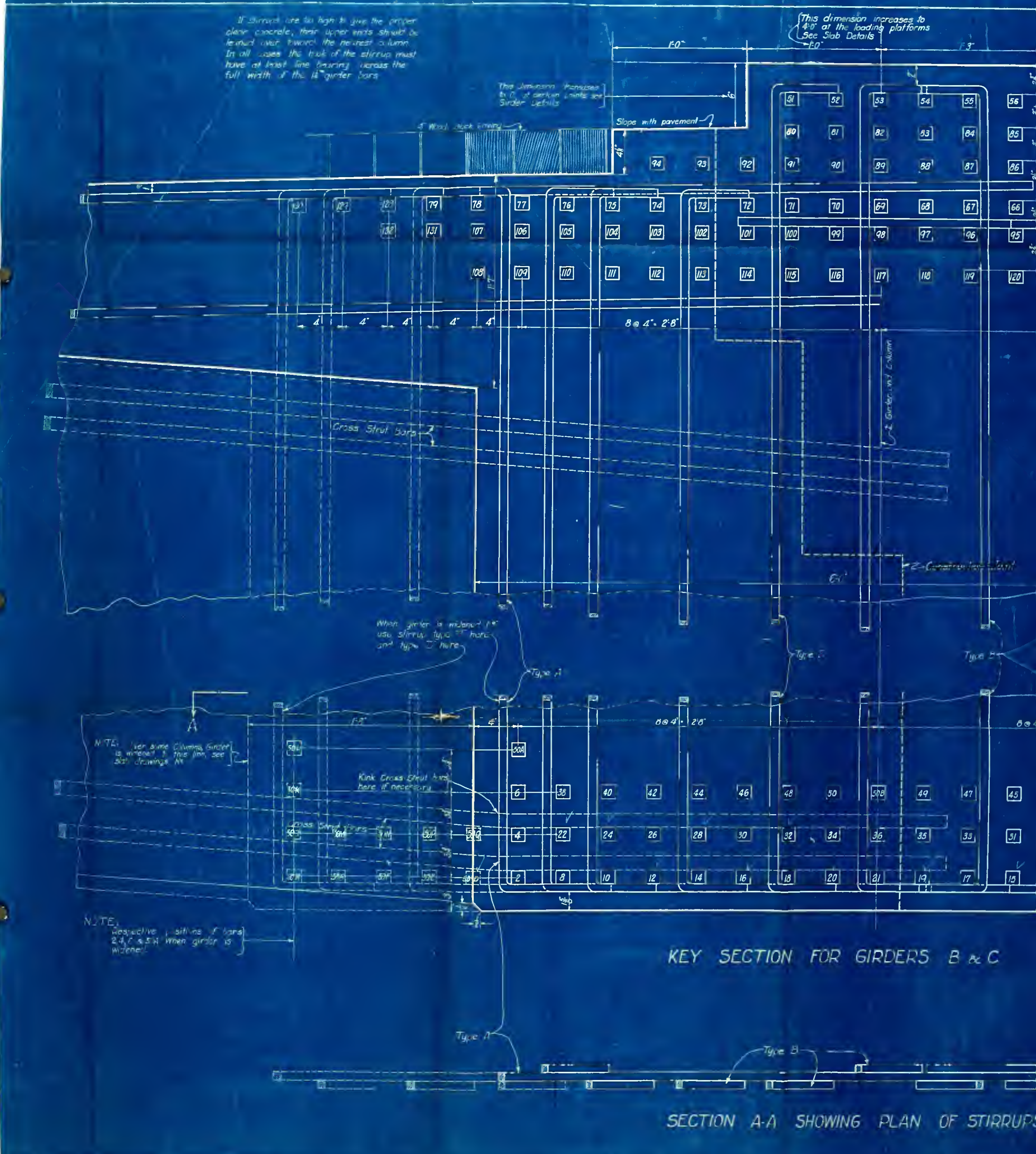
118 FT. VIADUCT

ROOSEVELT ROAD
(FORMERLY TWELFTH STREET)

WABASH AVENUE TO THE CHICAGO RIVER
SUPERSTRUCTURE
CROSS SECTION OF GIRDERS B & C

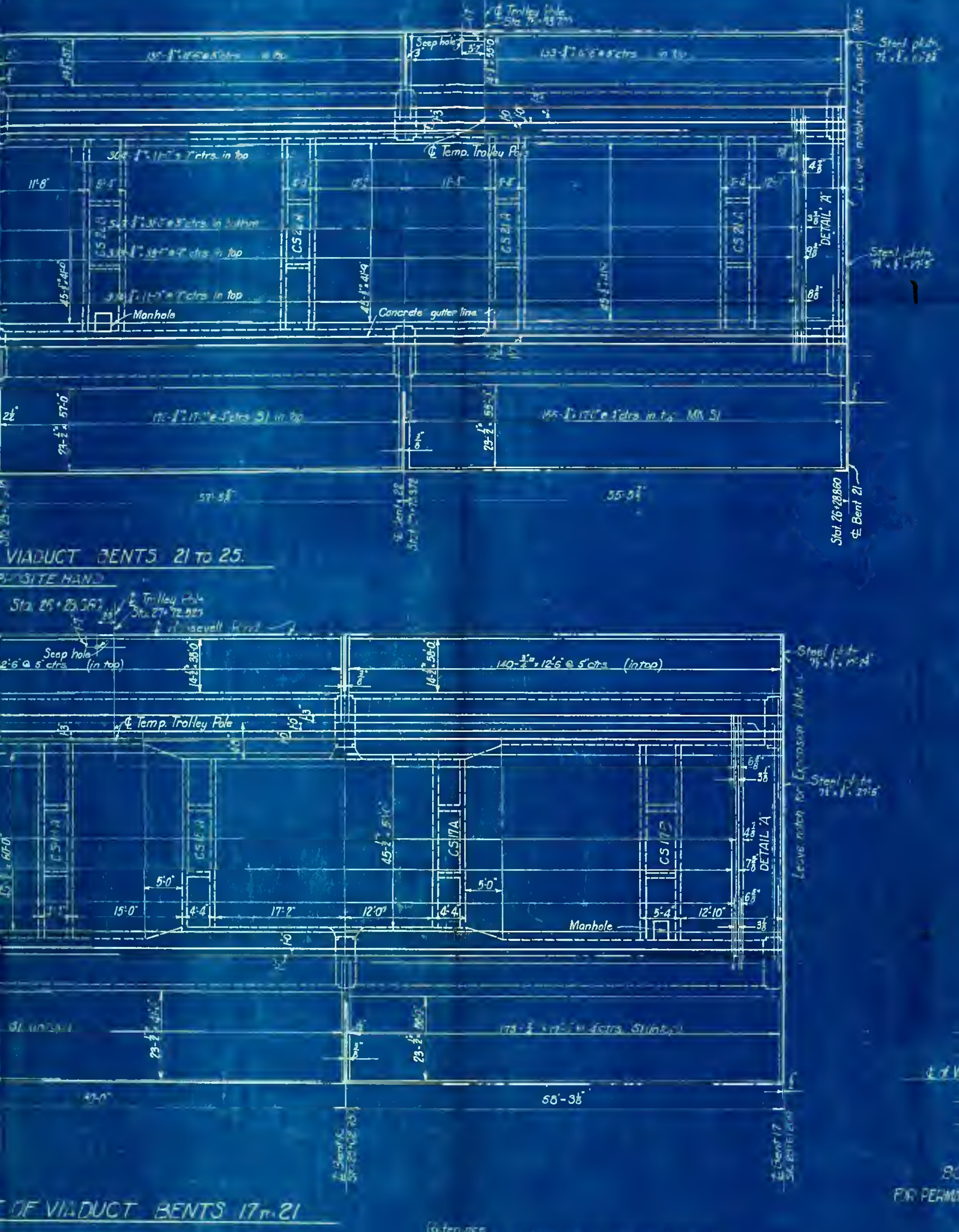
Drawn by R.G.F. & H.L.
Traced by H.R.B.
Checked by F.W.B.
7212

4E per 103

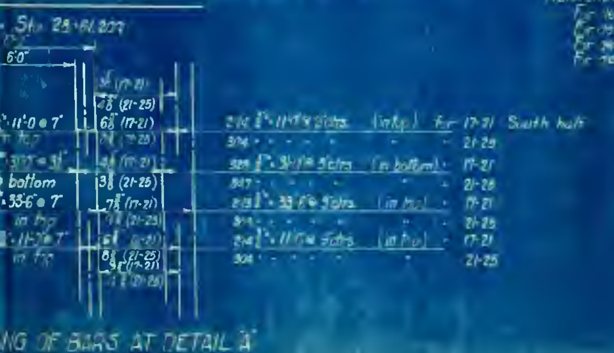

If Stirrups are too high to give the proper clear concrete, their upper ends should be leaned over toward the nearest column. In all cases the hook of the stirrup must have at least line bearing across the full width of the 1¼" girder bars
This dimension increases to 4'0" at the loading platforms See Slab Details
1'0"
1'0"
1'3"
This dimension increases to 1'... at certain points see Girder Details
Slope with pavement
4" Wood block paving
4½"
8 @ 4" = 2'8"
4"
Cross Strut Bars
℄ Girder and Column
℄ Construction Joint
6'1"
When girder is widened 1'8" use stirrup type "F" here and type "C" here
Type A
Type C
Type B
A
1'8"
8 @ 4" = 2'8"
NOTE; Over some Columns Girder is widened to this line, see Slab drawings
Kink Cross Strut bars here if necessary
Cross Strut Bars
NOTE; Respective positions of bars 2, 4, 6 & 50A when girder is widened
Type A
Type B

KEY SECTION FOR GIRDERS B & C

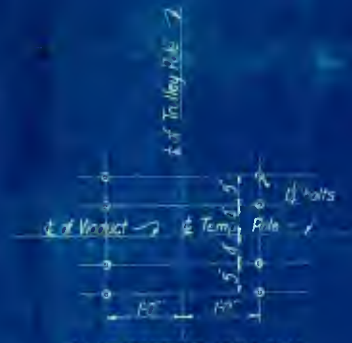
SECTION A-A SHOWING PLAN OF STIRRUPS



VIADUCT BENTS 21 TO 25.
OPPOSITE HAND
Seep hole
℄ Trolley Pole Sta. 26+83.???
133-¾" ?'6" @ 5" ctrs. in top
℄ Temp. Trolley Pole
304-¾" 11'-7" @ 7" ctrs. in top
54-¾" 35'-6" @ 5" ctrs. in bottom
33-¾" 38'-4" @ 7" ctrs. in top
314-¾" 11'-7" @ 7" ctrs. in top
Manhole
Concrete gutter line
CS 21 A
CS 21 A
CS 21 A
CS 21 A
DETAIL 'A'
Leave notch for Expansion Plate
Steel plate
Steel plate
171-¾" 11'-0" @ 5" ctrs. 31 in top
165-¾" 17'-0" @ 5" ctrs. in top. MK 31
11'-8"
57'-5⅜"
55'-5⅜"
℄ Bent 22
Sta. 26+28.860
℄ Bent 21
Sta. 26+25.387
℄ Trolley Pole Sta. 27+72.927
℄ Roosevelt Road
Seep hole
2'-6" @ 5" ctrs. (in top)
140-¾" 12'-6" @ 5" ctrs. (in top)
℄ Temp. Trolley Pole
CS 17 A
CS 17 A
CS 17 A
CS 17 A
DETAIL 'A'
Manhole
Leave notch for Expansion Plate
Steel plate
Steel plate
15'-0"
5'-0"
4'-4"
17'-2"
12'-0"
4'-4"
5'-0"
5'-4"
12'-10"
45'-2" 51'-0"
58'-3⅜"
℄ Bent 17
℄ Bent 18
PLAN OF VIADUCT BENTS 17 TO 21



Reference
For details of Cross section see dwg. 4E per 1/5
For details of Cross Struts see " " " " " 6/01
For details of Man-Holes " " " " " 1/4
For details of Drains " " " " " 1/3



BOLT SETTING DIAGRAM
FOR PERMANENT & TEMPORARY CASTING

CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING

118 FT. VIADUCT

ROOSEVELT ROAD

(FORMERLY TWELFTH STREET)

WABASH AVENUE TO THE CHICAGO RIVER

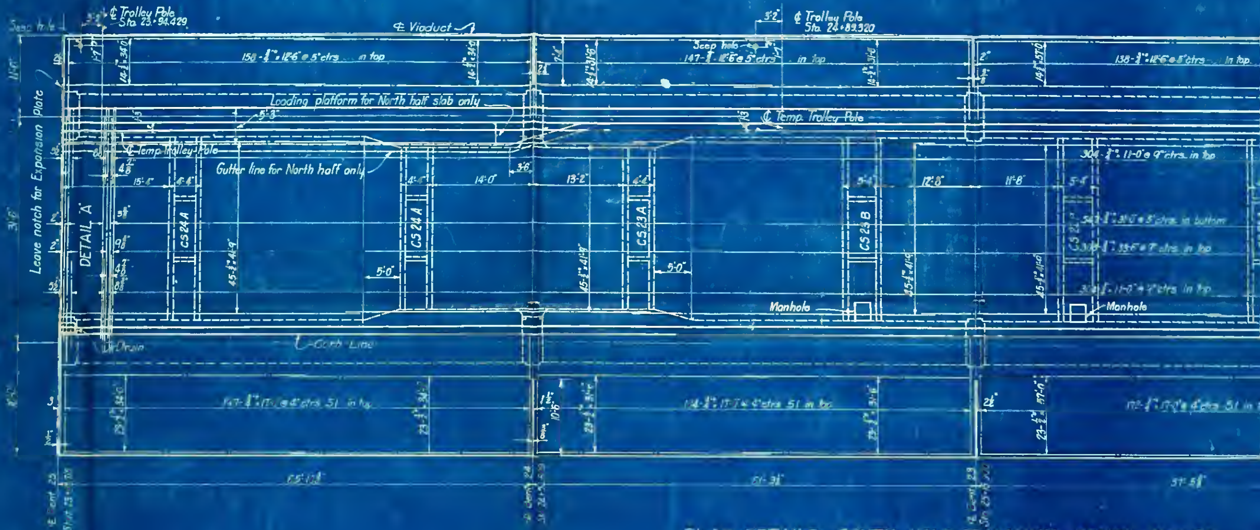
SUPERSTRUCTURE

SLAB DETAIL BENTS 17 TO 25

Scale ⅛"=1'-0"   March 19??

Drawn by ? & R.F.
Traced by ?
Checked by ?
Drawing No. 7214

4E per 5

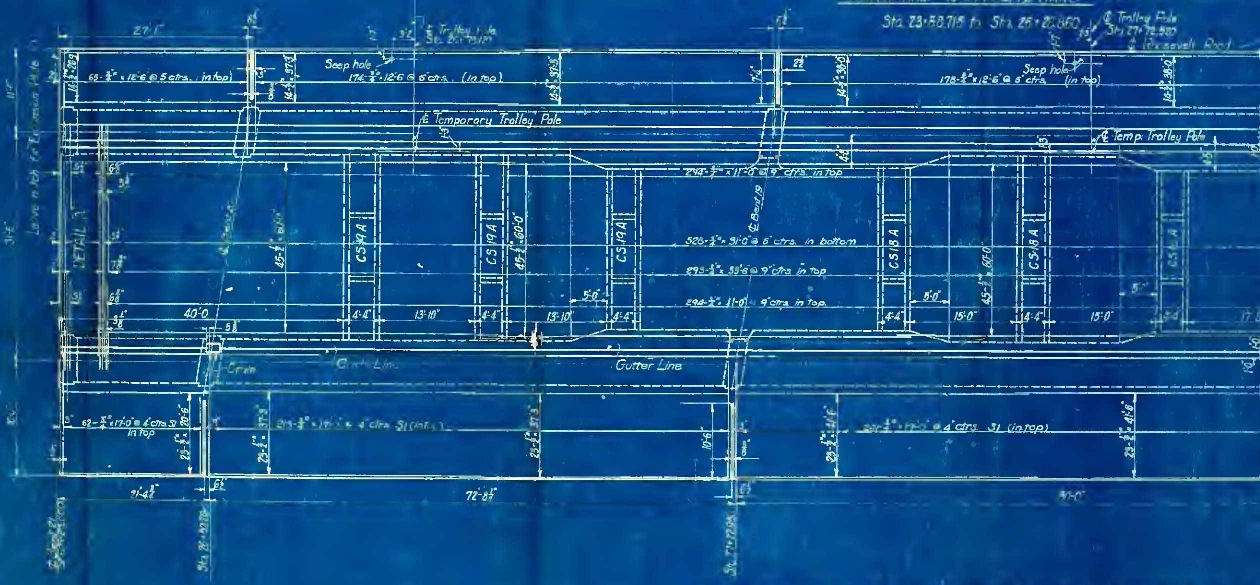


℄ Trolley Pole Sta. 23+94.429
℄ Viaduct
℄ Trolley Pole Sta. 24+89.320
Seep hole
158-¾" × 12'6" @ 5" ctrs. in top
147-¾" × 12'6" @ 5" ctrs. in top
138-¾" × 12'6" @ 5" ctrs. in top
Loading platform for North half slab only
℄ Temp. Trolley Pole
Gutter line for North half only
Leave notch for Expansion plate
DETAIL "A"
CS 24 A
CS 23 A
CS 23 B
Manhole
Drain
Curb Line
304-¾" × 11'0" @ 9" ctrs. in top
147-¾" × 17'0" @ 4" ctrs. S1 in top
134-¾" × 17'7" @ 4" ctrs. S1 in top
℄ Bent 25 Sta. 23+58.718
℄ Bent 24
℄ Bent 23
65'-10⅞"
51'-3⅞"
51'-5⅞"

SLAB DETAILS SOUTH HALF OF VIADUCT BENTS 21 TO 25.

NORTH HALF IS OPPOSITE HAND

Sta. 23+58.718 to Sta. 26+26.860



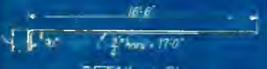
℄ Trolley Pole Sta. 27+72.980
℄ Roosevelt Road
Seep hole
68-¾" × 12'6" @ 5" ctrs. in top
174-¾" × 12'6" @ 5" ctrs. (in top)
178-¾" × 12'6" @ 5" ctrs. (in top)
℄ Temporary Trolley Pole
℄ Temp. Trolley Pole
Leave notch for Expansion Plate
DETAIL "A"
CS 19 A
CS 18 A
294-¾" × 11'0" @ 9" ctrs. in top
526-½" × 31'0" @ 6" ctrs. in bottom
293-½" × 33'6" @ 9" ctrs. in top
294-½" × 11'0" @ 9" ctrs. in top
℄ Bent 19
Drain
Curb Line
Gutter Line
62-¾" × 17'0" @ 4" ctrs. S1 in top
40'-0"
21'-4⅞"
72'-8⅞"

SLAB DETAILS SOUTH HALF OF VIADUCT BENTS 17 TO 2

Sta. 26+25.860 to Sta. 28+61.209

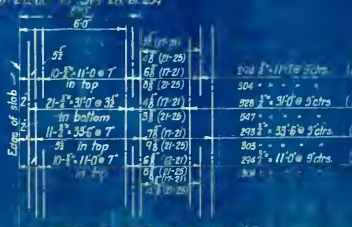
	No. of Bars	Size	Length	Remarks
BENTS 17 to 21 For South Half only	315	¾"	33'6"	
	570	"	31'0"	
	628	"	11'0"	
	558	"	12'6"	
	135	½"	60'0"	
	82	"	58'0"	
		"		
	46	"	41'6"	
	28	"	38'0"	
	74	"	37'3"	
	14	"	26'9"	
	23	"	20'6"	

	No. of Bars	Size	Length	Remarks
BENTS 21 to 25 For North and South Half.	650	¾"	33'6"	
	1178	"	31'0"	
	1296	"	11'0"	
	1152	"	12'6"	
	74	½"	57'0"	
	74	"	55'0"	
	540	"	41'9"	
	148	"	34'0"	
	148	"	31'6"	



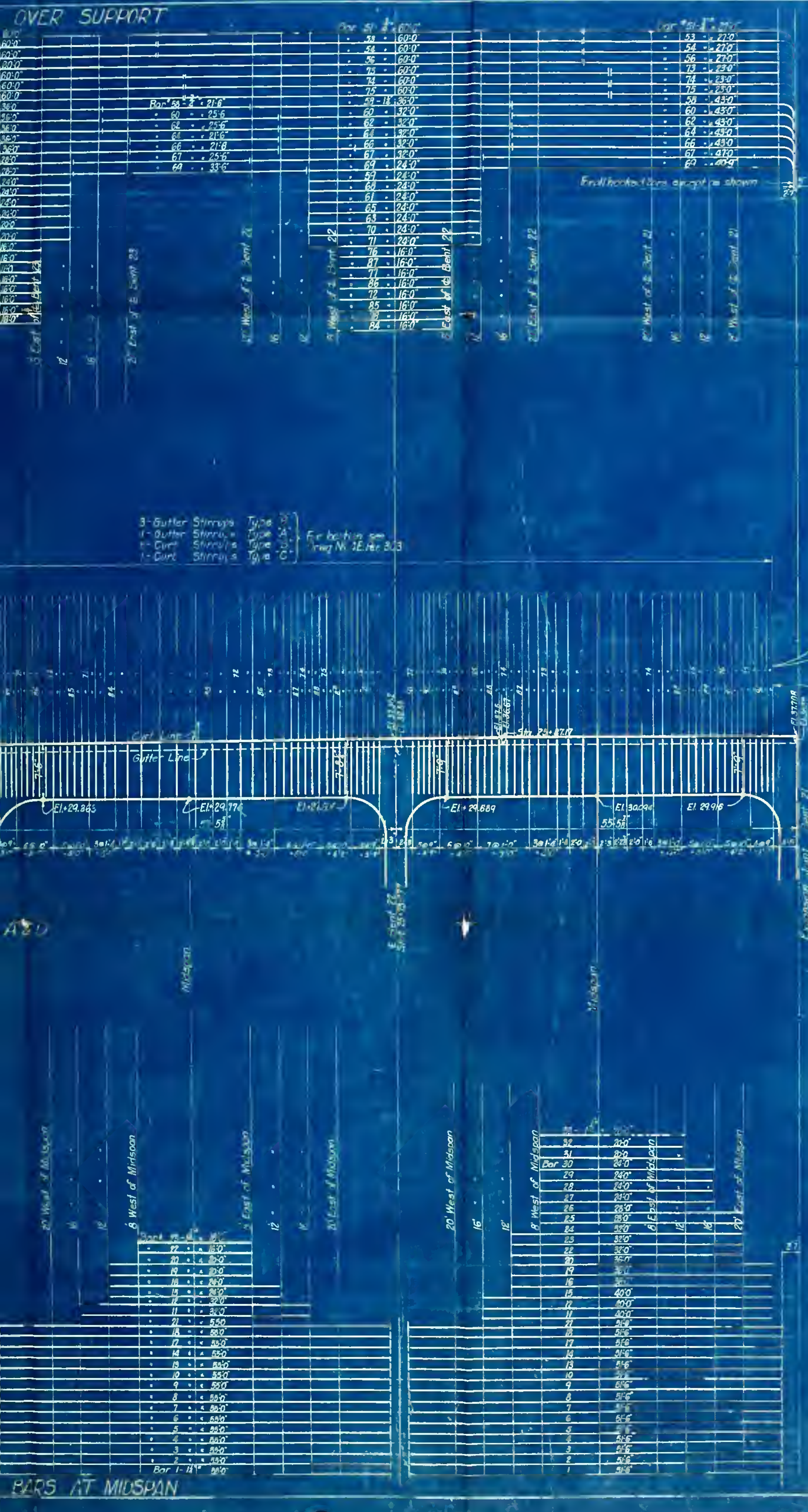
DETAIL of S1

1416 Reqd. between Bent 17-21 for South Half only.
1438 Reqd. " " 21-25 " North and South Half.



SPACING OF BARS AT DETAIL "A"

NOTE
For details of steel plates at expansion joints
see drwg. AE. Pur 158.
MIX 1:2:4



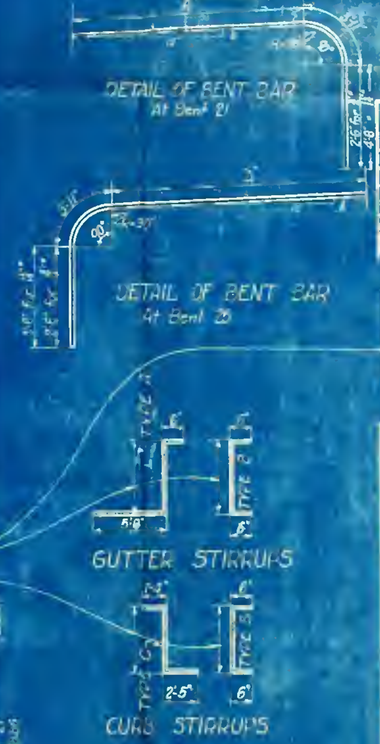
3-Gutter Stirrups Type B
1-Gutter Stirrup Type A
1-Curb Stirrup Type D
1-Curb Stirrup Type C
For bar bending see Drwg No 1E Per 303

LIST OF STIRRUPS

HEIGHT OF STIRRUP	NO. REQ'D. & LENGTH NEEDED		HEIGHT OF STIRRUP	NO. REQ'D & LENGTH NEEDED	
	TYPE 'C'	TYPE 'B'		TYPE 'B'	TYPE 'A'
91"	6 - 11'-4	36 - 8'-9"	77"		31 - 12'-5
90"	13 - 11'-3	78 - 8'-5	76"	39 - 7'-4	75 - 12'-4
89"	13 - 11'-2	[illegible]	75"	42 - 7'-3	14 - 12'-3
88"	14 - 11'-1	[illegible]	74"	63 - 7'-2	21 - 12'-2
87"	27 - 11'-0	162 - 8'-3	73"	99 - 7'-1	25 - 12'-1
86½"	12 - 10'-11	72 - 8'-2	72"	105 - 7'-0"	15 - 12'-0"
85"	14 - 10'-10	84 - 8'-1"	71"	102 - 6'-11"	18 - 11'-11"
84"	18 - 10'-9	108 - 8'-0"	70"	69 - 6'-10"	9 - 11'-10"
83"	12 - 10'-8	72 - 7'-11"	69"	69 - 6'-9"	9 - 11'-9"
82"	11 - 10'-7	66 - 7'-10"	68"	58 - 6'-8"	18 - 11'-8
81"	22 - 10'-6"	132 - 7'-9"	67"	12 - 6'-7"	4 - 11'-7"
80"	9 - 10'-5"	54 - 7'-8"		TYPE 'D'	
79"	8 - 10'-4"	48 - 7'-7"	73"	4 - 13'-7	
78"	9 - 10'-5	54 - 7'-6"	72"	10 - 13'-6	
77"	6 - 10'-2	69 - 7'-5"	71"	8 - 13'-5	
			70"	8 - 13'-4	
			69"	7 - 13'-3	

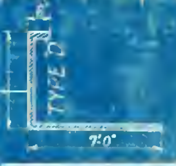
All stirrups ½" plain bars

These bar lists are for Girder 'D'
Bar lists for Girder A are the same



BENT MOMENT BAR LIST

№	Size	Length	'A'	Remarks
3	¾"	53'-6"	47'-1"	
3	"	49'-6"	43'-1"	
1	"	47'-0"	40'-7	
5	"	43'-0"	36'-7	



STRAIGHT MOMENT BAR LIST

№	Size	Length	Remarks
15	½"	62'-0"	
15	"	58'-9"	
15	"	55'-0"	
15	"	51'-6"	
6	"	51'-0"	
3	"	48'-0"	
5	"	44'-0"	
10	"	40'-0	
16	"	36'-0	
18	"	32'-0"	
18	"	28'-0"	
32	"	24'-0	
18	"	20'-0"	
28	"	16'-0"	
14	¾"	60'-0"	
3	"	65'-3"	
4	"	51'-8"	
1	"	47'-3"	
8	"	40'-0"	
6	"	38'-0"	
1	"	33'-6"	
4	"	29'-3	
4	"	27'-0"	
3	"	25'-6"	
3	"	23'-0"	
6	"	21'-6"	

NOTE
Bars are not to lap at splices except as shown
Bars must have end bearing on each other, but their ends need not be squared to secure full bearing
For Cross Section of Girder see Drwg № 1E per 303
For arrangement of Corbel Bars see Drwg № 1E per 104
For Detail of Slab see Drwg № 1E per 5
For Cross Section of Slab see Drwg № 1E per 105
Mix 1:2:4

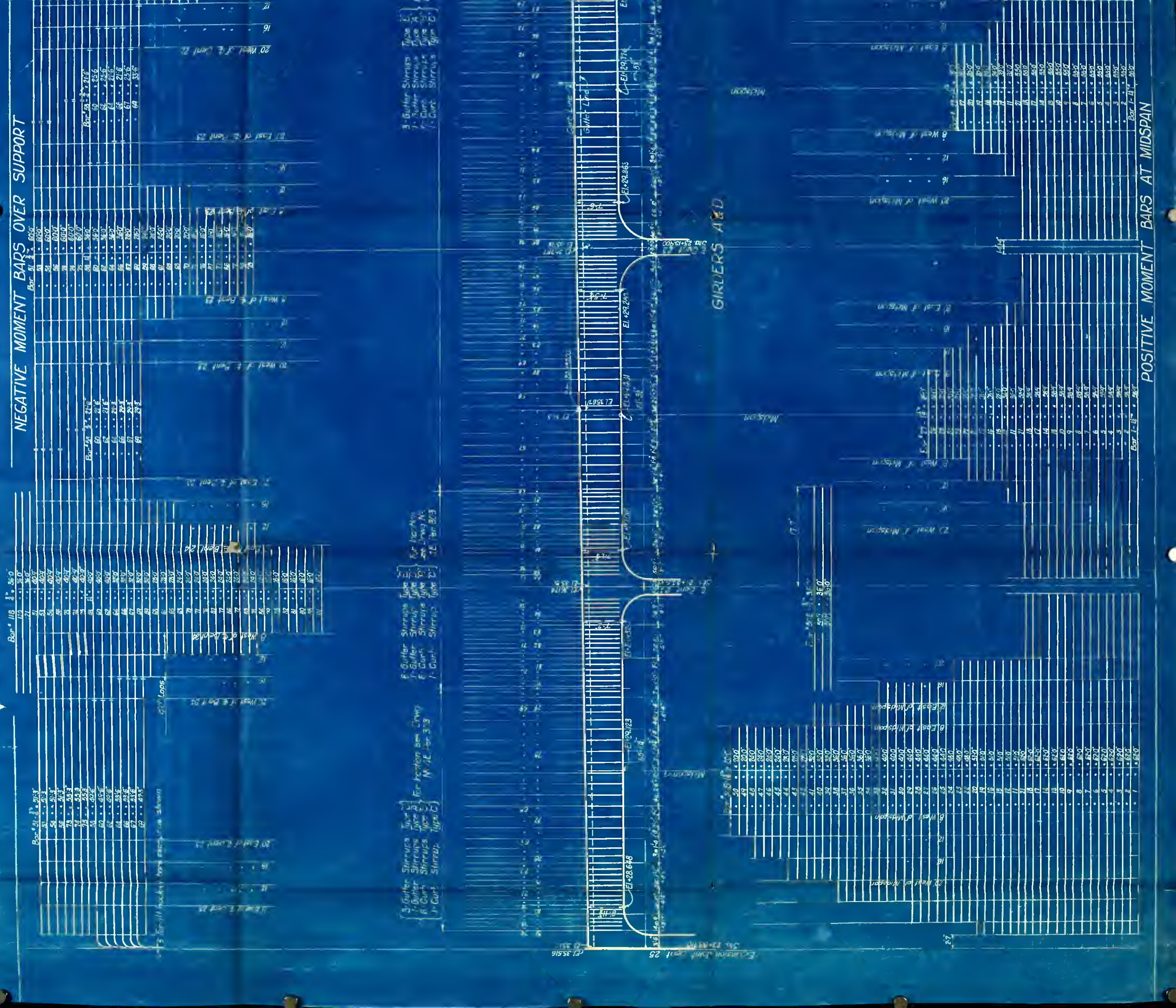
Reinforcing Steel - Plain Bars Structural Grade

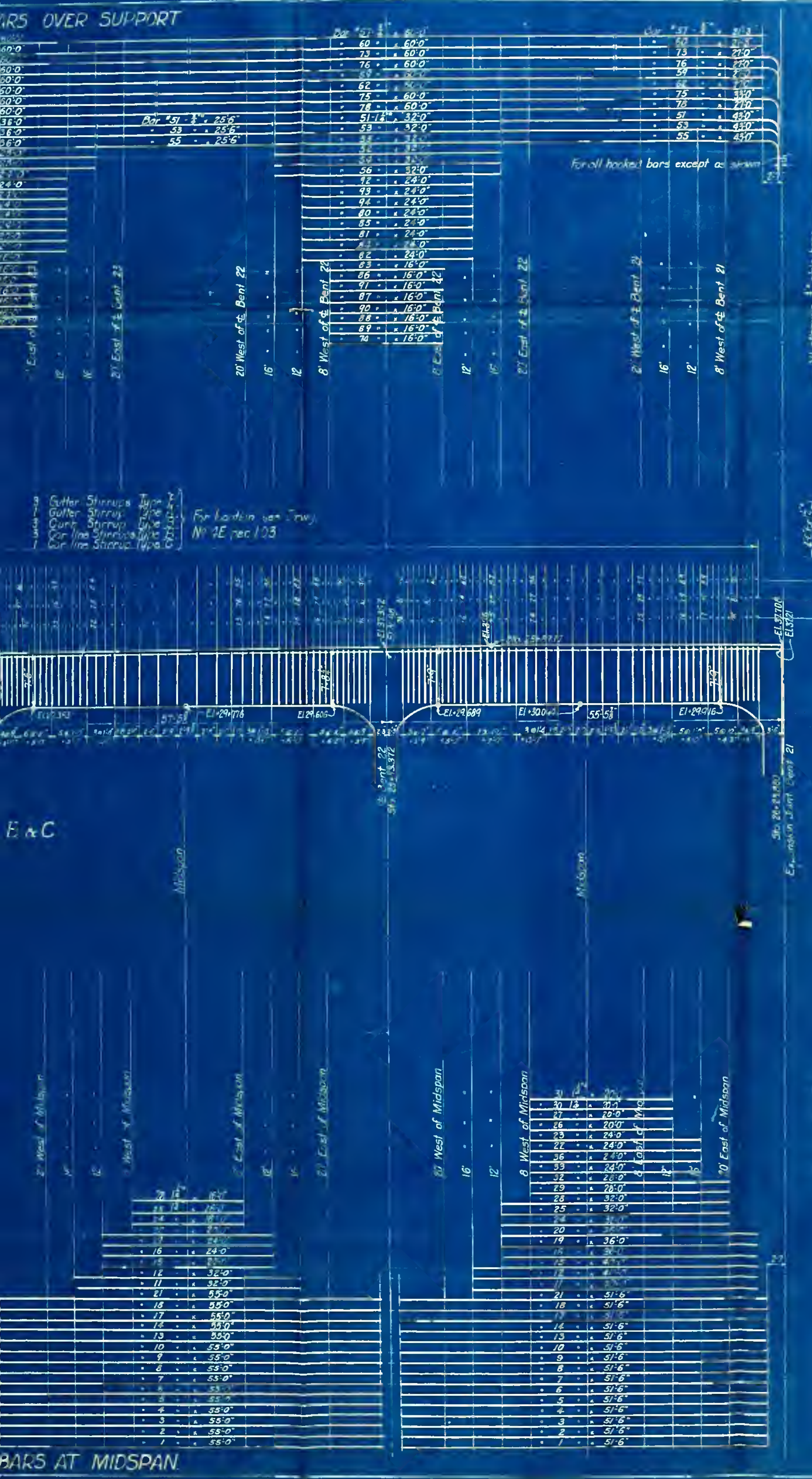
CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING
DIVISION OF BRIDGES

118 FT. VIADUCT

ROOSEVELT ROAD
FORMERLY TWELFTH STREET

WABASH AVENUE TO THE CHICAGO RIVER
SUPERSTRUCTURE
DETAIL OF GIRDERS A & D BENTS 21 TO 25


NEGATIVE MOMENT BARS OVER SUPPORT
GIRDERS A & D
POSITIVE MOMENT BARS AT MIDSPAN
Midspan
Expansion Joint Bent 25



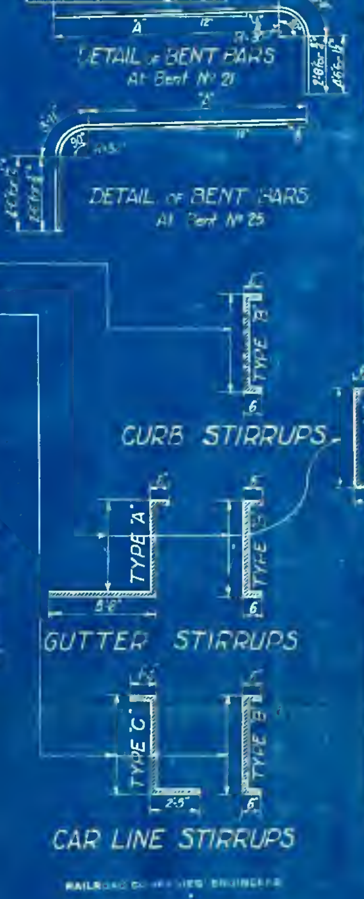
All stirrups ¾" plain bars

LIST OF STIRRUPS

HEIGHT OF STIRRUP	Nº REQUIRED & LENGTH BAR NEEDED Type "A"	Type "B"	Type "C"	Type "D"
65		30 6'-5"	10 9'-2"	
66		27 6'-6"	9 9'-3"	
67		24 6'-7"	8 9'-4"	
68		45 6'-8"	16 9'-5"	
69		48 6'-9"	16 9'-6"	
70		33 6'-10"	11 9'-7"	
71	5 11'-11"	57 6'-11"	14 9'-8"	
72	19 12'-0"	114 7'-0"	19 9'-9"	
73	8 12'-1"	108 7'-1"	12 9'-10"	8 13'-7"
74	8 12'-2"	132 7'-2"	20 9'-11"	8 13'-8"
75	16 12'-3"	162 7'-3"	22 10'-0"	8 13'-9"
76	15 12'-4"	132 7'-4"	11 10'-1"	9 13'-10"
77	26 12'-5"	174 7'-5"	14 10'-2"	4 13'-11"
78	22 12'-6"	132 7'-6"	13 10'-3"	
79	12 12'-7"	60 7'-7"		
80	14 12'-8"	87 7'-8"		
81	12 12'-9"	84 7'-9"		
82		33 7'-10"		
83		42 7'-11"		
84		57 8'-0"		
85		36 8'-1"		
86		72 8'-2"		
87		54 8'-3"		
88		33 8'-4"		
89		42 8'-5"		
90		39 8'-6"		

STRAIGHT MOMENT BAR LIST

Nº	Size	Length	Remarks
15	1¼"	62'-0"	
15	"	58'-9"	
15	"	55'-0"	
15	"	51'-6"	
5	"	51'-0"	
4	"	48'-0"	
5	"	44'-0"	
7	"	40'-0"	
16	"	36'-0"	
19	"	32'-0"	
15	"	28'-0"	
43	"	24'-0"	
13	"	20'-0"	
34	"	16'-0"	
16	¾"	60'-0"	
4	"	51'-0"	
4	"	60'-0"	
4	"	48'-0"	
4	"	41'-0"	
6	"	36'-0"	
3	"	27'-0"	
3	"	25'-6"	
5	"	21'-3"	



NOTE
Bars are not to lap at splices except as shown.
Bars must have end bearing on each other, but their ends need not be squared to secure full bearing.
For Cross Section of Girders see Drwg. Nº 4E per 103
For arrangement of Corbel Bars see Drwg. Nº 4E per 104
For Detail of Slab see Drwg. Nº 4E per 5
For Cross-Section of Slab, see Drwg. Nº 4E per 105
MIX 1:2:4
Reinforcing Steel {Plain Bars, Structural Grade

BENT MOMENT BAR LIST

Nº	Size	Length	"A"	Remarks
3	1¼"	55'-6"	47'-1"	
3	¾"	43'-0"	36'-7"	
1	"	33'-0"	26'-7"	
2	"	27'-0"	20'-7"	

These bar lists are for Girder "C". Bar lists for Girder "B" are the same, except as noted.

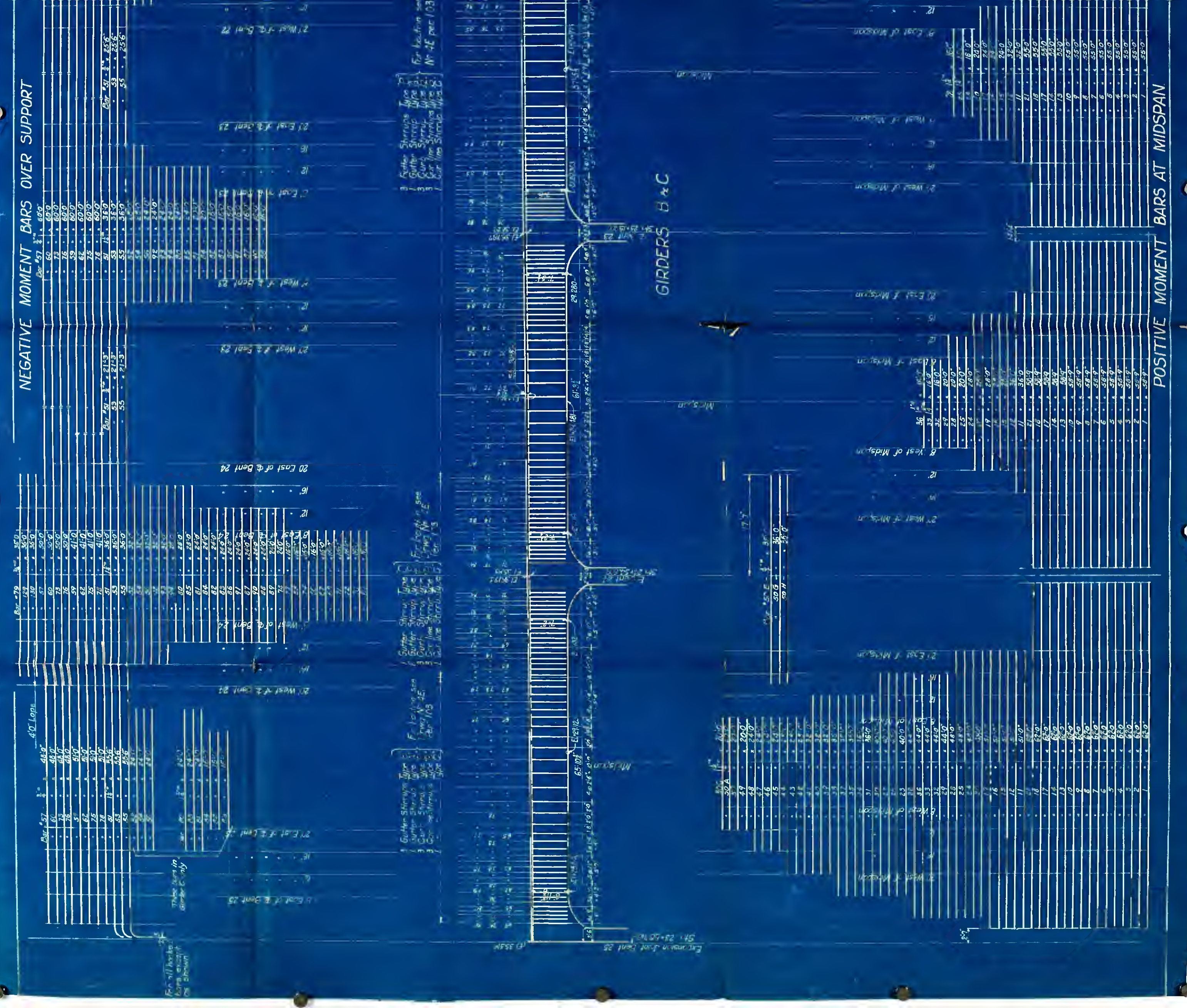
CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING
DIVISION OF BRIDGES

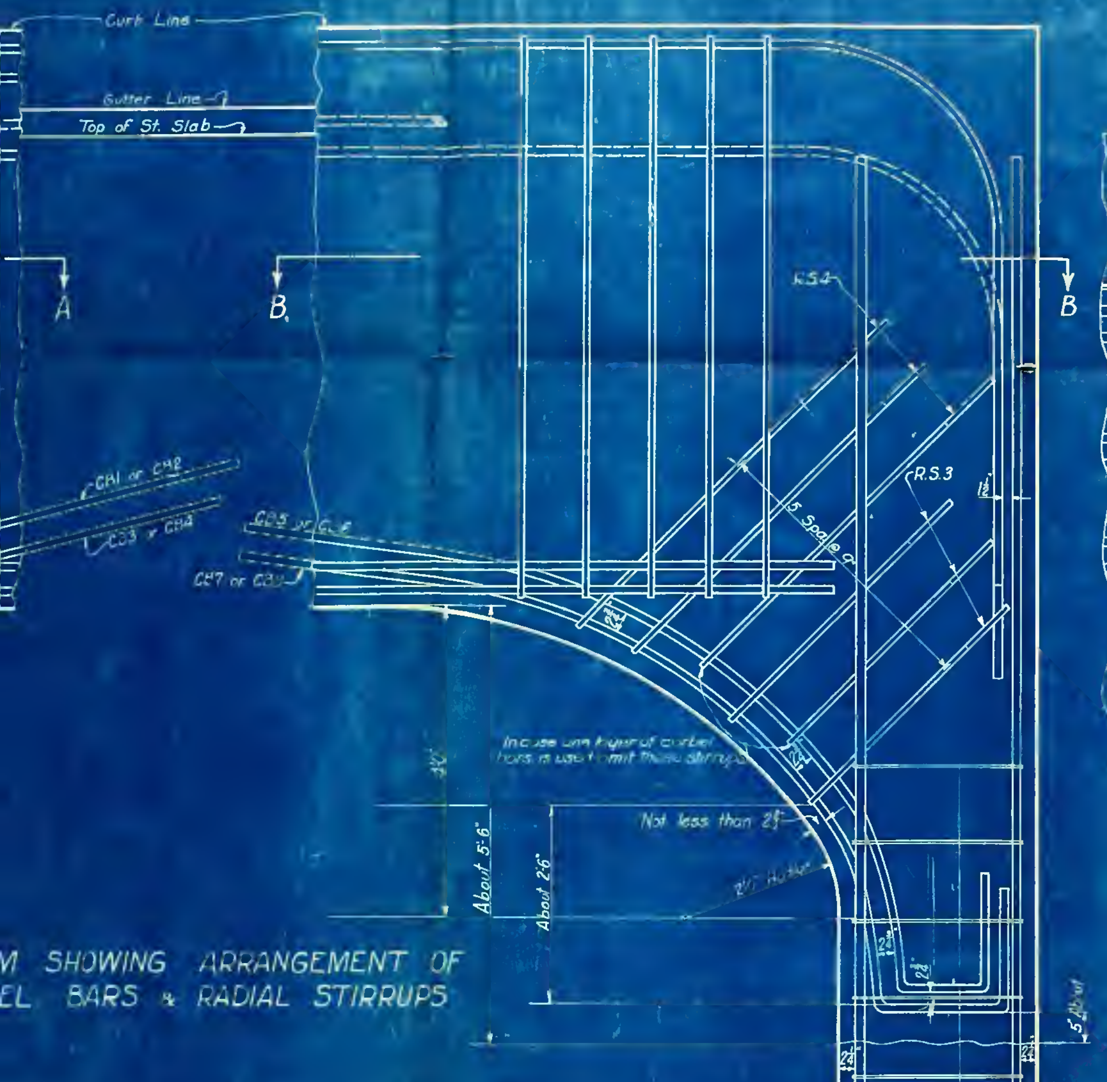
118 FT. VIADUCT

ROOSEVELT ROAD
(FORMERLY TWELFTH STREET)
FROM
WABASH AVENUE TO THE CHICAGO RIVER
SUPERSTRUCTURE
DETAIL of GIRDERS B & C. BENTS 21 to 25

Scale ⅛"=1'-0" March 1927
Drawn by H.G.I.
Traced by A.E.G. Drawing No. 71.91
Checked by Fife No. 25

4E per 107

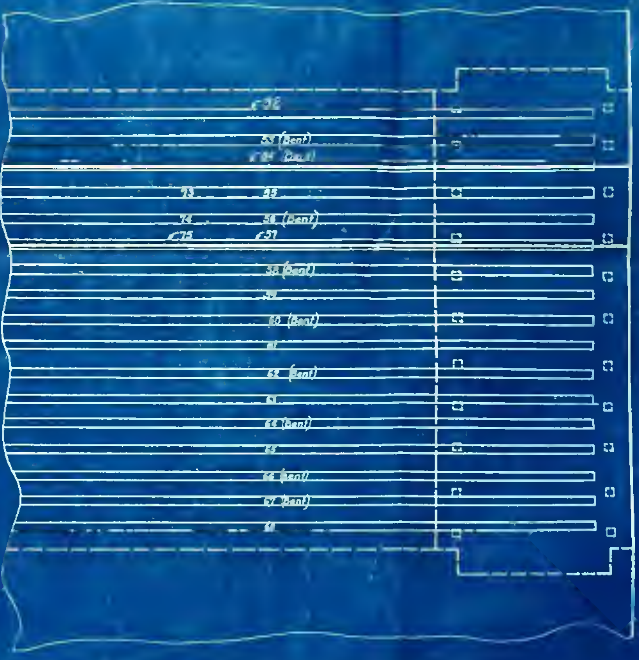

NEGATIVE MOMENT BARS OVER SUPPORT
GIRDERS B & C
POSITIVE MOMENT BARS AT MIDSPAN
27 West of ℄ Bent 22
27 East of ℄ Bent 23
27 West of ℄ Bent 23
27 West of ℄ Bent 23
20 East of ℄ Bent 24
20 West of ℄ Bent 24
27 East of ℄ Bent 25
6 East of ℄ Bent 25
Midspan
8 East of Midspan
21 West of Midspan
27 West of Midspan
21 East of Midspan
8 West of Midspan
27 East of Midspan
4'0" Laps
These bars in Girder C only
For all hooked bars except as shown
Gutter Stirrups
Curb Stirrups
Car line Stirrups
For location see Dwg No. 4E per 103
Exp. Jnt. in Slab ℄ Bent 25
GIRDERS



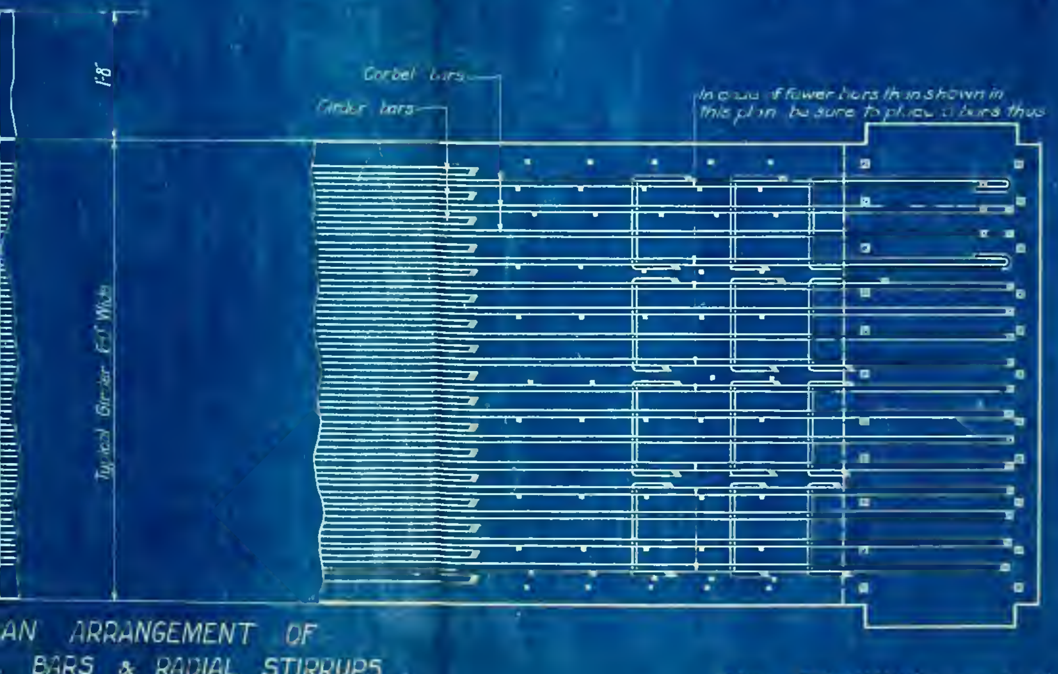
M SHOWING ARRANGEMENT OF
EL BARS & RADIAL STIRRUPS

END COLUMN (ELEVATION)

NOTE
This layout is made for 16 C-road bars in each of 2 layers, if column schedule calls for less than 16 bars lay bars in one layer.



PLAN ARRANGEMENT OF NEGATIVE
MOMENT BARS – END COL.
GIRDERS A & D
GIRDERS B & C SIMILAR


1'-8"
Typical Girder 6'-0" Wide
Corbel bars
Girder bars
In case of fewer bars than shown in this plan be sure to place ☐ bars thus

AN ARRANGEMENT OF
BARS & RADIAL STIRRUPS

END COLUMN (SECTION 'B-B')

RAILROAD COMPANIES' ENGINEERS

Drawn Hugh E. Young
Approved Thos. G. Pihlfeldt
Approved K. F. [illegible]

CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING
DIVISION OF BRIDGES

118 FT. VIADUCT

ROOSEVELT ROAD
(FORMERLY TWELFTH STREET)

WABASH AVENUE TO THE CHICAGO RIVER
SUPERSTRUCTURE
ARR'G'T of STEEL in COLUMN HEAD
Scale Dec.
Traced by H.E.Y.
Checked by Shea
Drawing No. 7281

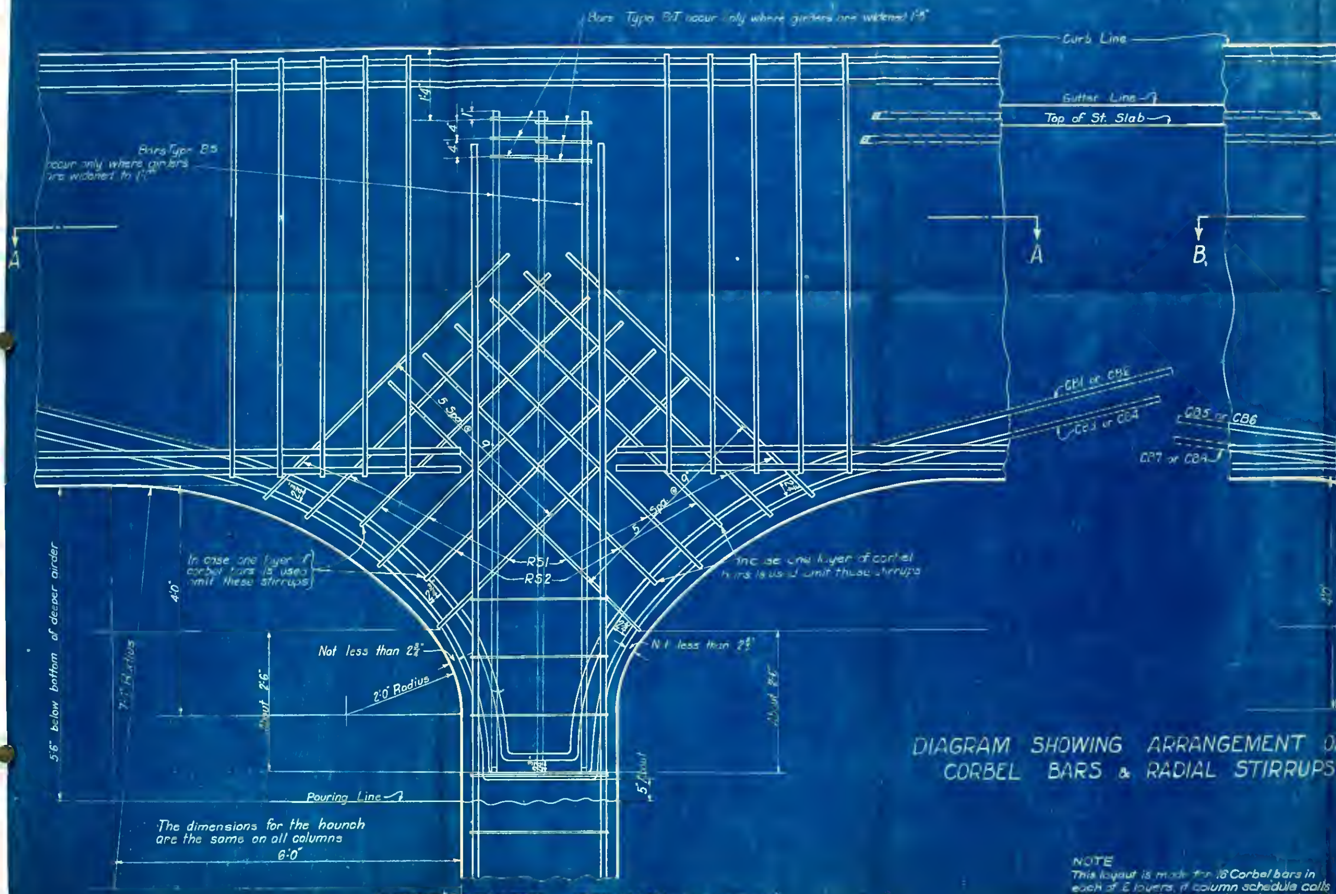
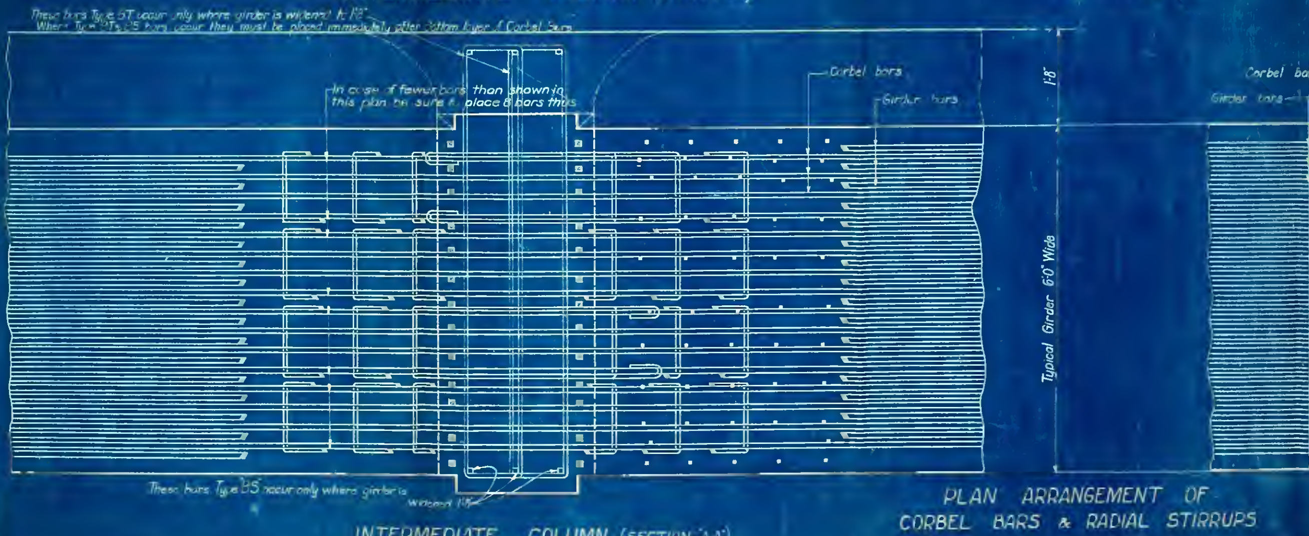

Bars Type BT occur only where girders are widened 1'8"
Curb Line
Gutter Line
Top of St. Slab
Bars Type B5 occur only where girders are widened to 1'8"
1'4"
4"
4"
A
A
B
5 Spa. @ 9"
5 Spa. @ 9"
CB1 or CB2
CB3 or CB4
CB5 or CB6
CB7 or CB8
R51
R52
In case one layer of corbel bars is used omit these stirrups
In case one layer of corbel bars is used omit these stirrups
4'0"
7'0" Radius
5'6" below bottom of deeper girder
Not less than 2¾"
Not less than 2¾"
2'0" Radius
About 2'6"
About 2'6"
5" About
Pouring Line
The dimensions for the haunch are the same on all columns
6'0"

DIAGRAM SHOWING ARRANGEMENT OF CORBEL BARS & RADIAL STIRRUPS

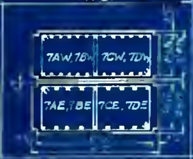
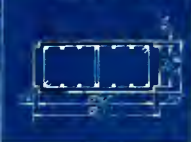
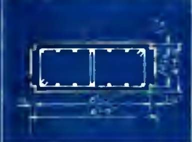
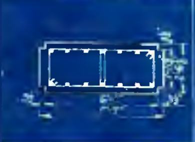
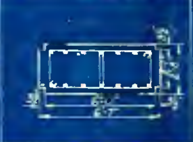
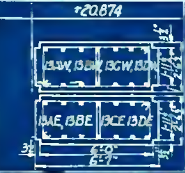
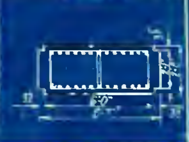
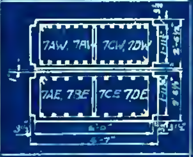
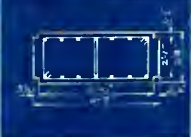
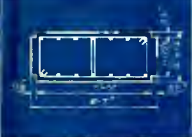
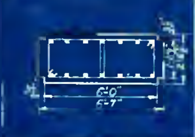
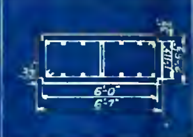
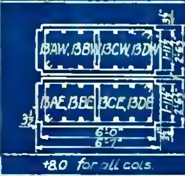
NOTE
This layout is made for 16 Corbel bars in each of 2 layers, if column schedule calls for less than 16 bars lay bars in one layer.

INTERMEDIATE COLUMN (ELEVATION)

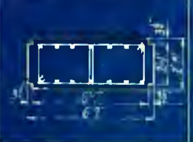
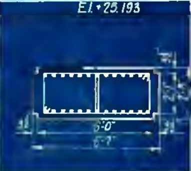
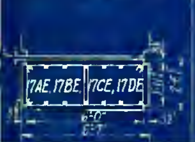
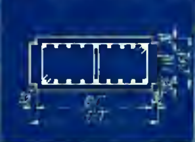
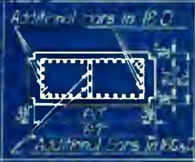
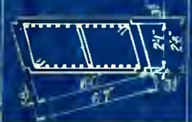
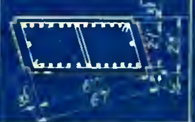

These bars Type 5T occur only where girder is widened to 1'8"
When Type 5T & 25 bars occur they must be placed immediately after bottom layer of Corbel bars
In case of fewer bars than shown in this plan be sure to place 8 bars thus
Corbel bars
Girder bars
Corbel bars
Girder bars
1'8"
Typical Girder 6'0" Wide
These bars Type B5 occur only where girder is widened 1'8"

INTERMEDIATE COLUMN (SECTION "A-A")

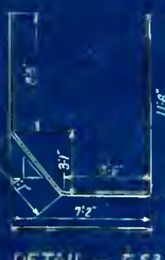
PLAN ARRANGEMENT OF CORBEL BARS & RADIAL STIRRUPS

COLUMN NUMBERS	7AE, 7BE, 7CE, 7DE. 7AW, 7BW, 7CW, 7DW.	8A, 8B, 8C, 8D.	9AE, 9BE, 9DE, 9CE.	12AW, 12BW, 12CW, 12DW.	12½A, 12½B, 12½C, 12½D	13AE, 13BE, 13CE, 13DE 13AW, 13BW, 13CW, 13DW.	14A, 14B, 14C, 14D.
Elev. of Curb	+25.75 at Sta 33+19.162	+27.33 at Sta 32+72.50	+28.685 at Sta 32+32.884	+30.415 at Sta. 31+26.135	+31.704 at Sta. 30+93.55	+32.31 at Sta. 30+70.77	+33.31 at Stat. 30+22.22
Elev. of Gutter At ℄ of Col. for Single Cols. " " Exp. Jt. - Double "	Col. A & D +24.90 Col. B & C 25.23	Col. A & D +26.50 Col. B & C +26.83	Col. A & D 28.17 Col. B & C. +28.50	Col. A & D +29.955 Col. C & B +30.285	Col. A & D +30.993 Col. C & B +31.323	Col. A & D +31.48 Col. C & B. +31.81	COL. A & D +33.08 COL. B & C +33.41
Elev. of Pouring line	+14.27	+15.91	+17.225	+19.544	+20.287	+20.874	El. +24.006
SECTION AT POURING LINE							
TIES	20-¼"⌀ × 10'-11" MK CT2 12" c.c. for each col.	24-¼"⌀ × 11'-2" MK CT1 12" c.c.	20-¼"⌀ × 10'-11" MK CT2 12" c.c.	24-¼"⌀ × 10'-11" MK CT2 12" c.c.	32-¼"⌀ × 11'-2" MK CT1 12" c.c.	40-¼"⌀ × 10'-11" MK CT2 12" c.c. for each col.	40-¼"⌀ × 11'-2" MK CT1 12" c.c. for each col.
MIX	1-2-4	1-2-4	1-2-4	1-2-4	1-2-4	1-2-4	1-2-4
VERTICAL BARS	26-1¼"⌀ × 16'-0" } for cols. 7AW, 7BW, 7CW, 7DW. 18-1¼"⌀ × 16'-0" for 7AE, 7BE, 7CE, 7DE	14-1¼"⌀ × 17'-6"	14-1¼"⌀ × 15'-3"	14-1"⌀ × 17'-0"	14-1"⌀ × 22'-0"	14-1"⌀ × 22'-6" for 13AE, 13BE, 13CE, 13DE. 14-1¼"⌀ × 22'-6" for 13AW, 13BW, 13CW, 13DW.	26-1¼"⌀ × 24'-0"
SECTION 1'-0" ABOVE TOP OF SUB-COLUMN							Same as at pouring line
Probable El. Top of sub-col.	+8.00 for all cols.	+8.00 for all cols.	+11.50 for all cols.	+11.5 for all cols.	+8.0 for all cols.	+8.0 for all cols.	+8.0 for all cols.
See Drg 4E Per. 169 CORBEL BARS	8-1"⌀ × 16'-0" CB8 In West Col. 14-1"⌀ × 16'-0" CB8 - East -	10-1"⌀ × 27'-6" CB4	12-1¼"⌀ × 16'-0" CB7.	10-1¼"⌀ × 16'-0" CB7	12-1"⌀ × 27'-6" CB4	8-1"⌀ × 16'-0" CB8 in East Col. 12-1¼"⌀ × 16'-0" CB7 in West Col.	16-1¼"⌀ × 27'-6" MK CB3
RADIAL STIRRUPS	8-¾"⌀ × 10'-6" RS3 } In each Col. 8-½"⌀ × 13'-9" RS4 }	32-¾"⌀ - 15'-3" RS1	8-¾"⌀ × 10'-6" RS3 8-½"⌀ × 13'-9" RS4	8-¾"⌀ × 10'-6" RS3 8-½"⌀ × 13'-9" RS 4	32-¾"⌀ × 15'-3" RS1	[illegible] In east Column [illegible] In west Column Use in place of RS3 and RS4	32-¾"⌀ - 13'-3" MK RS1a in place of bars MK RS1
REMARKS	In col. 7AW & 7DW insert holders for downspouts support See Drg. No. 4E Per. 149					In col. 13AW & 13DW insert holders for downspouts supports See drwg. 4E Per. 149	6-¾"⌀ × 22'-9" MK BT1 3-¾"⌀ × 22'-2" MK BS1 See Drwg. 4E Per. 192 for Details

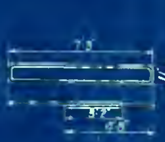
NOTE:- For columns between expansion joints in bents 9 and 12 see details of State St. Arch

COLUMN NUMBERS	15A, 15B, 15C, 15D	16A, 16B, 16C, 16D	17AE, 17BE, 17CE, 17DE	17AW, 17BW, 17CW, 17DW	18A, 18B, 18C, 18D.	19A	19B
Elev. of Curb	+35.98 at Stat. 29+63.929	+37.55 of Stat. 29+16.00	+38.375 - at Stat. 28+61.209	+38.375 at Sta. 28+61.209	+38.579 at Sta 28+02.95	+38.249 at Sta. 27+37.05	+38.223 at Sta. 27+31.885
Elev. of Gutter At ℄ of Col. for Single Cols. " " Exp. Jt. - Double "	Col. A & D +35.15 Col. B & C +35.48	Col. A & D +36.72 Col. B & C +37.05	Col. A & D +37.545 Col. B & C +37.875	Col. A & D +37.545 Col. B & C +37.875	Col. A & D +37.749 Col. B & C +38.079	Col. A +37.419	Col. B +37.723
Elev. of Pouring Line.	El. +24.623	El. +25.193	El. +25.733	El. +25.733	El. +25.409	El. +25.035	El. +25.006
SECTION AT POURING LINE							
TIES	32-⅜"⌀ × 11'-2" MK CT1 12" c.c. for each column	32-⅜"⌀ × 11'-2" MK CT1 12" c.c. for each col.	46-⅜"⌀ × 10'-11" MK CT2 12" c.c. for each col.	46-⅜"⌀ × 10'-11" Mk. CT2 12" c.c. for each col.	44-⅜"⌀ × 11'-2" Mk. CT1 12" c.c. for each col.	42-⅜"⌀ × 11'-3" Mk. CT1A 12" c.c.	42-⅜"⌀ × 11'-3" Mk. CT1A 12" c.c.
MIX	1-2-4	1-2-4	1-2-4	1:2:4	1:2:4	1:2:4	1:2:4
VERTICAL BARS	14-1"⌀ × 26'-9"	26-1¼"⌀ × 28'-3"	14-1¼"⌀ × 29'-0"	18-1"⌀ × 29'-0"	26-1¼"⌀ × 29'-3" for 18A, B 28-1¼"⌀ × 29'-3" for 18C 34-1¼"⌀ × 29'-3 for 18D	26-1¼"⌀ × 29'-0"	28-1¼"⌀ × 29'-0"
SECTION 1'-0" ABOVE TOP OF SUB-COLUMN	Same as at pouring line	Same as at pouring line	Same as at pouring line	Same as at pouring line	Same as at pouring line	Same as at pouring line	Same as at pouring line
Probable El. Top of sub-col.	+7.5 for all col.	+7.5 for all col.	+7.5 for all col.	+7.5 for all cols.	+7.5 for all cols.	+8.0	+8.0
See Drwg 4E Per. 169 CORBEL BARS	12-1¼"⌀ × 27'-6" MK CB 3	16-1¼"⌀ × 27'-6" MK CB 3 4-1"⌀ × 27'-9" MK CB 2	8-1¼"⌀ × 16'-0" MK CB 7	8-1¼"⌀ × 16'-0" Mk CB7	12-1¼"⌀ × 27'-6" Mk CB-3	8-1¼"⌀ × 27'-6" Mk. C.B3	8-1¼"⌀ × 27'-6" Mk C.B3
RADIAL STIRRUPS	32-¾"⌀ × 15'-3" MK RS1	32-¾"⌀ × 15'-3" MK RS1 8- " × 14'-3" MK RS2	8-¾"⌀ × 10'-6" MK RS-3 8- " × 13'-9" MK RS-4	8-¾"⌀ × 10'-6" Mk RS3 8-½"⌀ × 13'-9" Mk RS4	32-¾"⌀ × 15'-3" Mk. RS1	32-¾"⌀ × 15'-3" Mk. R.S.1	32-¾"⌀ × 15'-3" Mk. R.S.1
REMARKS	6-¾"⌀ × 19'-6" MK BT2 3- " " × 24'-6" MK BS2 See Drwg N° 4E Per. 192 for details				6-¾"⌀ × 19'-6" MK BT2 3-¾"⌀ × 28'-6" Mk BS3 In Cols. 18C & 18D only	6-¾"⌀ - 19'-6" Mk BT2 3-¾"⌀ × 28'-6" Mk BS3	6-¾"⌀ × 19'-6" Mk BT2 3-¾"⌀ × 28'-6" Mk BS3

NOTE:- In Col. 19A insert holders for Hydrant Pipe Supports See Drwg. N° 4E Per.



DETAIL of BS3
¾"⌀ × 28'-6"



DETAIL of BT2
¾"⌀ × 19'-6"

Hugh E. Young

113 FT. VIADUCT

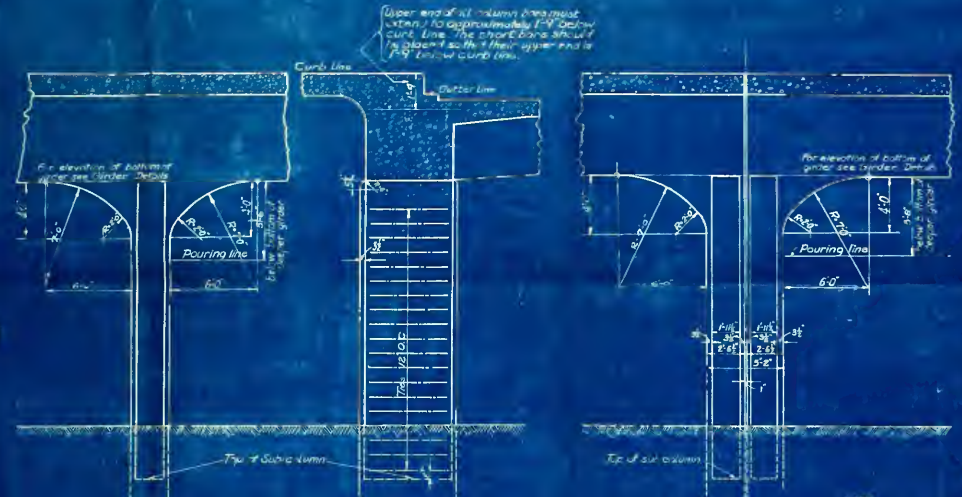
ROOSEVELT ROAD

SUPERSTRUCTURE
COLUMN SCHEDULE

R.V.K.
J.A.H.

7252

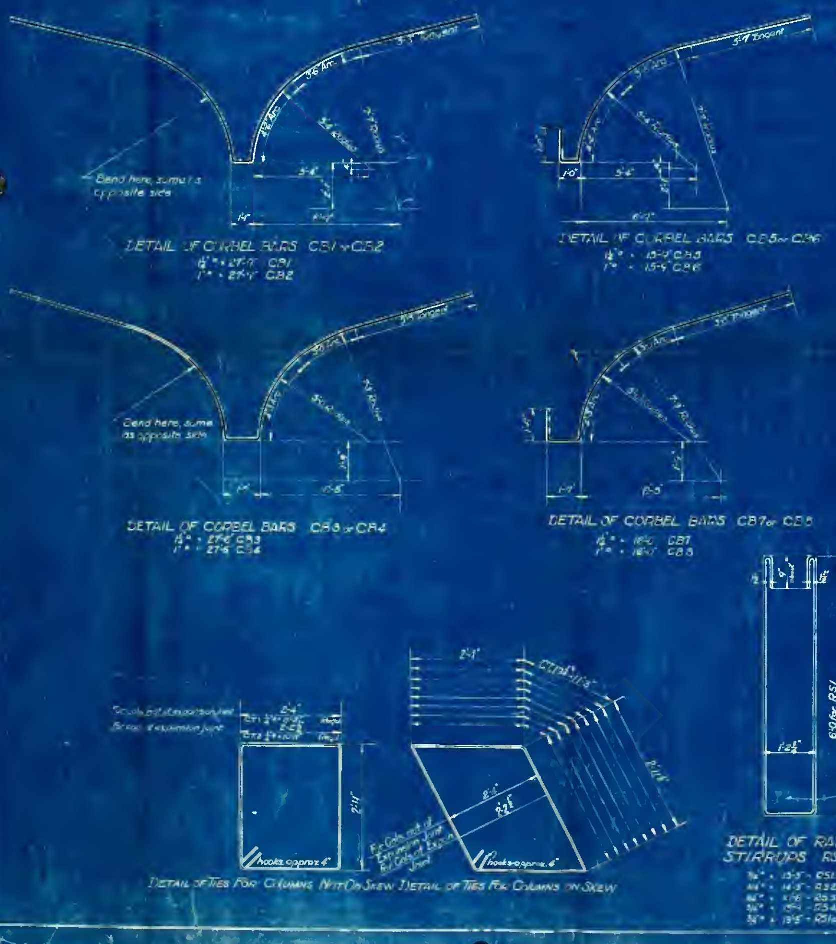
4E PER.102


Upper end of all column bars must extend to approximately 1'-9" below curb line. The short bars should be placed so that their upper end is 1'-9" below curb line.
Curb line
Gutter line
For elevation of bottom of girder see Girder Details
Pouring line
Ties 12 O.C.
Top of Sub column
For elevation of bottom of girder see Girder Details
Pouring line
6'-0"
4'-0"
R=7'0"
below bottom of deeper girder
Top of sub column

NOTE

Between the top of the subcolumn and the bottom of the girder the columns must not in anyway be denied freedom to bend thru the soft earth. Concrete pavement should be kept from them by at least an inch of mastic

Clear distance between Vertical Bars and face of column is 2"



DETAIL OF CORBEL BARS CB1 or CB2
1½" • 27'-11" CB1
1" • 27'-4" CB2

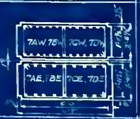
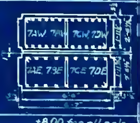
DETAIL OF CORBEL BARS CB5 or CB6
1¼" • 15'-9" CB5
1" • 15'-9" CB6

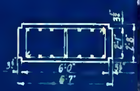
DETAIL OF CORBEL BARS CB3 or CB4
1¼" • 27'-6" CB3
1" • 27'-6" CB4

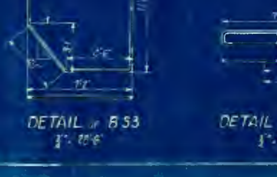
DETAIL OF CORBEL BARS CB7 or CB8
1¼" • 16'-0" CB7
1" • 16'-0" CB8

DETAIL OF TIES FOR COLUMNS NOT ON SKEW DETAIL OF TIES FOR COLUMNS ON SKEW

DETAIL OF RADIAL STIRRUPS RS1, RS2, RS3, RS4

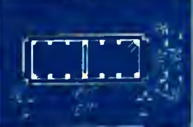
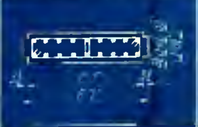
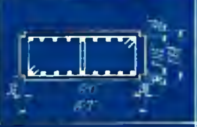
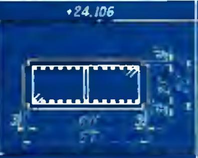
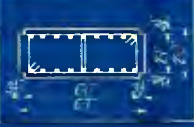
COLUMN NUMBERS		7AE, 7BE, 7CE, 7DE, 7AW, 7BW, 7CW, 7DW.	
Elev. of Curb		+25.73 at Sta 33+19.162	
Elev. of Gutter At ℄ of Col. for Single Cols. - Exp. Jt. - Double		Col. A&D +24.90	Col. B&C 25.23
Elev. of Pouring line		+14.27	
SECTION AT POURING LINE			
TIES		20-¼"ɸ × 10'-11" MKCT2 2" o.c. for each col.	
MIX		1-2-4	
VERTICAL BARS		26-1¼" • 16'-0" for cols 7AW, 7BW, 7CW, 7DW. 18-1¼" × 16'-0" for 7AE, 7BE, 7CE, 7DE	
SECTION 1'-0" ABOVE TOP OF SUB-COLUMN			
Probable El. Top of subcol.		+8.00 for all cols.	
SEE Drg 4E Per 169	CORBEL BARS	8-1" • 16'-0" CB8 In West Col. 14-1" × 16'-0" CB8 - East -	
	RADIAL STIRRUPS	8-½" • 10'-6" RS3 8-½" × 13'-9" RS4 In each Col	
REMARKS		In col. 7AW & 7DW insert holders for downspouts support See Drg. No. 4E Per. 149	

COLUMN NUMBERS		15A, 15B, 15C, 15D	
Elev of Curb		+35.98 at Stat. 29.+63.929	
Elev. of Gutter At ℄ of Col. for Single Cols. - Exp. Jt. - Double		Col. A & D +35.15	Col. B & C +35.48
Elev. of Pouring Line.		El. +24.623	
SECTION AT POURING LINE			
TIES		42-½" • 11'-2" MK CT1 12" o.c. for each column	
MIX		1-2-4	
VERTICAL BARS		14-1" × 26'-9"	
SECTION 1'-0" ABOVE TOP OF SUB-COLUMN		Same as at pouring line	
Probable El. Top of sub col.		+7.5 for all col.	
SEE DRWG 4E PER. 169	CORBEL BARS	12-1¼" • 27'-6" MK CB 3	
	RADIAL STIRRUPS	32-½" • 15'-3" MK RS1	
REMARKS		6-½" • 19'-6" MK BT2 3- " " • 21'-6" MK BS2 See Drwg No 4E Per. 192 for details	



DETAIL of B 53
½" • 15'-6"

DETAIL

20D	21AE 21BE	21CE 21DE	21AW-21BW-21CW-21DW	22A-22B-22C-22D	23A-23B-23C-23D	24A-24B-24C-24D	25AE-25BE-25CE-25DE	
at Sta 26+50.26	37.708 at Sta 26+28.86	37.708 at Sta. 26+28.86	37.708 at Sta. 26+28.86	37.362 at Sta. 25+73.372	36.787 at Sta 25+15.90	36.174 at Sta. 24+54.609	35.516 at Sta. 23+88.718	
Col. D +36.986	Col. A +36.878 Col B +37.208	Col C +37.208 Col. D +36.878	Col. A & D +36.878 Col. B & C +37.208	Col. A & D +36.532 Col. B & C +36.862	Col. A & D +35.957 Col. B & C. +36.287	Col. A & D +35.50 Col. B & C +35.83	Col. A & D +35.186 Col. B & C +35.516	
+24.542	+24.511	+24.505	+24.416	+24.106	+23.780	+23.432	+23.148	
as of 21A						Same as 22 (A-B-C-D)	Same as 21 (AW-BW-CW-DW)	
1'-3" Mk CT19 12" o.c.	40-¼"ø × 10'-11" Mk C.T.2 12" o.c. for each col.	40-¼"ø × 9'-2" Mk C.T.3 12" o.c. for each col.	40-¼"ø × 10'-11" Mk C.T.2 12" o.c. for each col.	42-¼"ø × 11'-2" Mk. C.T.1 12" o.c. for each col.	40-¼"ø × 11'-2" Mk C.T.1 12" o.c. for each col.	38-¼"ø × 11'-2" Mk C.T.1 12" o.c. for each col.	40-¼"ø × 10'-11" Mk. C.T.2 12" o.c. for each col.	
1¼" × 28'-3"	14-1"ø × 27'-6"	18-1¼"ø × 27'-6"	18-1¼"ø × 27'-6"	26-1¼"ø × 28'-3"	18-1"ø × 27'-0"	26-1¼"ø × 26'-0"	18-1¼"ø × 26'-9"	
+8.0	+8.50	+8.50	+8.50	+7.50	+8.0	+8.50	+7.00	
27'-6" Mk. C.B.3	8-1¼" × 16'-0" Mk. C.B.7	8-1¼" × 16'-0" Mk. C.B.8	12-1¼"ø × 16'-0" Mk. C.B.7	12-1¼"ø × 27'-6" Mk C.B.3	8-1"ø × 27'-6" Mk C.B.4	16-1¼"ø × 27'-6" Mk CB3	16-1¼"ø × 16'-0" Mk. CB7	
15'-3" Mk R.S.1	8-¾"ø × 10'-6" Mk R.S.3 8-¾"ø × 13'-9" Mk R.S.4	8-¾" × 13'-9" Mk. R.S.4 8-¾" × 8'-3" Mk R.S.3a RS3a in place of RS3	8-¾"ø × 10'-6" Mk R.S.3 8-¾"ø × 13'-9" Mk. R.S.4	32-¾"ø × 15'-3" Mk R.S.1	32-¾"ø × 15'-3" Mk R.S.1	32-¾"ø × 15'-3" Mk. R.S.1	8-¾"ø × 10'-6" Mk. R.S.3 8-¾"ø × 13'-9" Mk R.S.4	
20D insert for downspouts see Drwg. Nº 49.	In column 21AE insert holders for downspouts supports see Drwg Nº 4E per 149					6-¾"ø × 19'-6" Mk BT2 3 ¾" × 27'-6" Mk. BS4	In columns 25AE & 25DE insert holders for downspouts supports, see Drwg. Nº 4E per 149	

OF BT2

NOTES

Mix 1:2:4

For details called for but not shown here see Drwg Nº 4E Pen 102

Clear distance between vertical rods and face of column is 2"

Correct ____ Engineer of Bridge Design

Approved ____ Engineer of Bridges

Approved ____ City Engineer

Approved ____ Commissioner of Public Works

RAILROAD COMPANIES' ENGINEERS

Approved ____ Chief Engineer [illegible]

Approved ____ Chief Engineer [illegible]

Approved ____ [illegible]

Approved ____ Chief Engineer [illegible]

Approved ____ Chief Engineer [illegible]

Approved ____

CITY OF CHICAGO
DEPARTMENT OF PUBLIC WORKS
BUREAU OF ENGINEERING
DIVISION OF BRIDGES

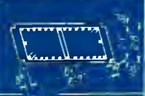
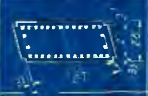
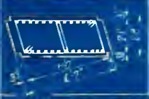
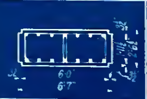
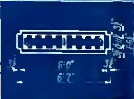
118 FT VIADUCT
FOR
ROOSEVELT ROAD
(FORMERLY TWELFTH STREET)
FROM
WABASH AVENUE TO THE CHICAGO RIVER

SUPERSTRUCTURE
COLUMN SCHEDULE

Scale 1"=1'0" 1919

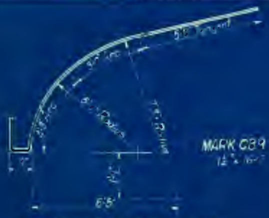
Drawn by L.S.T. Drawing Nº 7253
Traced by H.F.B.
Checked by Blume File Nº ____

4E per 104

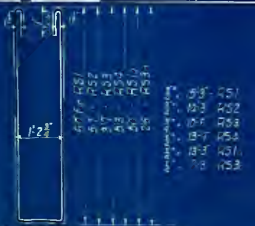
COLUMN NUMBERS	19C	19D	20A	20B	20C	20D	21AE 21BE	21CE 21DE
ELEV. OF CURB	38.205 at Sta. 27+28.115	38.179 at Sta 27+22.95	37.886 at Sta 26+64.36	37.861 at Sta. 26+59.195	37.841 at Sta. 26+55.425	37.816 at Sta 26+50.26	37.708 at Sta. 26+28.86	37.708 at Sta. 26+28.86
ELEV. OF GUTTER At ℄ of Col. for Single Cols " " Exp.Jt. " Double "	Col. C +37.705	Col. D +37.349	Col. A +37.056	Col. B +37.361	Col. C +37.341	Col. D +36.986	Col. A +36.878 Col. B +37.208	Col. C +37.208 Col. D +36.878
ELEV. OF POURING LINE	El +24.984	El.+24.955	El.+24.623	El.+24.593	El.+24.572	El.+24.542	+24.511	+24.505
SECTION				Same as at 20A	Same as at 20A	Same as at 20A		
TIES	42-½"ø × 11'3" Mk. CT.19 12" o.c.	42-½"ø × 11'3" Mk. CT.19 12" o.c.	42-½"ø × 11'3" Mk. CT.19 12" o.c.	42-½"ø × 11'3" Mk C.T.19 12" o.c.	42-½" × 11'3" Mk. CT.19 12" o.c.	42-½"ø × 11'3" Mk CT19 12" o.c.	40-½"ø × 10'11" Mk C.T.2 12" o.c. for each col	40-½"ø × 9'2" Mk C.T.3 12" o.c. for each col.
VERTICAL BARS	28-1¼"ø × 28'6"	30-1¼"ø × 28'6"	26-1¼"ø × 28'3"	26-1¼"ø × 28'3"	26-1¼"ø × 28'3"	26-1¼"ø × 28'3"	14-1"ø × 27'6"	18-1¼"ø × 27'6"
Probable Elev. Top of Subcol.	+8.0	+8.0	+8.0	+8.0	+8.0	+8.0	+8.50	+8.50
SEE DRWG 4E per 169 CORBEL BARS	8-1¼"ø × 27'6" Mk. CB3	8-1¼"ø × 27'6" Mk. C.B.5	12-1¼"ø × 27'6" Mk. C.B.3	12-1¼"ø × 27'6" Mk. C.B.3	12-1¼"ø × 27'6" Mk. C.B.3	12-1¼"ø × 27'6" Mk. C.B.3	8-1¼"ø × 16'0" Mk. C.B.7	8-1¼"ø × 16'0" Mk. C[illegible]
RADIAL STIRRUPS	32-¾"ø × 15'3" Mk. R.S.1	32-¾"ø × 15'3" Mk. R.S.1	32-¾"ø × 15'3" Mk. R.S.1	32-¾"ø × 15'3" Mk. R.S.1	32-¾"ø × 15'3" Mk. R.S.1	32-¾"ø × 15'3" Mk R.S.1	8-¾"ø × 10'6" Mk RS.3 8-¾"ø × 13'9" Mk RS4	8-¾"ø × 13'9" Mk RS4 8-¾"ø × 8'3" Mk RS3a RS3a in place of RS3
REMARKS	6-¾"ø × 19'6" Mk. BT2 3-¾"ø × 28'6" Mk. BS3	6-¾"ø × 19'6" Mk BT2 3-¾"ø × 28'6" Mk BS3				In column 20D insert holders for downspouts supports, see Drwg. Nº 4E par. 149.	In column 21AE insert holders for downspouts supports see Drwg. Nº 4E per 149	

COLUMN NUMBERS								
ELEV. OF CURB								
ELEV. OF GUTTER At ℄ of Col. for Single Column " " Exp.Jt. " Double "								
ELEV. of POURING LINE								
SECTION								
TIES								
VERTICAL BARS								
Probable Elev. Top of Subcol.								
SEE DRWG 4E per 169 CORBEL BARS								
RADIAL STIRRUPS								
REMARKS								

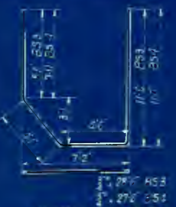
COLUMN NUMBERS								
ELEV. OF CURB								
ELEV. OF GUTTER At ℄ of Col. for Single Col. " " Exp.Jt. " Double "								
ELEV. of POURING LINE								
SECTION								
TIES								
VERTICAL BARS								
Probable Elev. Top of Subcol.								
SEE DRWG 4E per 169 CORBEL BARS								
RADIAL STIRRUPS								
REMARKS								



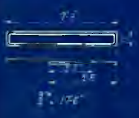
DETAIL OF CORBEL BAR



DETAIL OF RADIAL STIRRUP



DETAIL OF BS



DETAIL OF BT2